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Titre :

SCHÉMAS NUMÉRIQUES POUR LES MODÈLES DE TURBULENCE STATISTIQUES EN UN POINT.

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Le manuscrit est organisé de la manière suivante :

Le premier chapitre est un résumé de l'ensemble des travaux effectués. Il propose une introduction aux modèles physiques abordés pour le traitement de la turbulence et propose un modèle mathématique qui sert de cadre à cette thèse. Puis les discrétisations par les méthodes d'éléments finis non-conformes et de volumes finis sont décrites ainsi que l'algorithme complet. Enfin trois sous-problèmes d'analyse numérique sont définis, permettant d'étudier les propriétés du schéma numérique proposé. Ce sont les trois axes d'étude qui constituent les chapitres suivants.

Le deuxième chapitre est une version étendue d'un article publié dans le cadre du symposium Finite Volumes for Complex Applications V qui s'est tenu à Aussois en Juin 2008 et détaille la construction d'un schéma pour la résolution des équations de bilans couplées des échelles turbulentes, préservant la positivité de la solution.

Le troisième chapitre est un article soumis à CALCOLO, qui décrit la construction et l'analyse, à partir notamment des résultats usuels de l'approximation des équations de Stokes par éléments finis non-conformes et des résultats de convergence des schémas volumes finis avec second membre irrégulier, du schéma éléments finis/volumes finis dans le cas de la résolution d'un modèle de turbulence simplifié basé sur une viscosité turbulente de type “longueur de mélange”.

Le dernier chapitre est un article soumis à Mathematics of Computation et décrit l'approximation par une méthode de volumes finis d'une équation de transport caractéristique des modèles de turbulence RANS au premier ordre. Le problème envisagé est l'équation de convection–diffusion scalaire avec donnée L^1, dont la régularité du second membre provient de l'expression de la production turbulente. Ce travail permet d'effectuer un premier pas vers l'analyse du problème complet instationnaire.

Enfin, deux annexes sont proposées : la première est un article soumis à International Journal for Finite Volumes auquel j'ai pu apporter une petite contribution, qui aborde l'analyse d'un schéma de volumes finis pour le modèle de diffusion radiative $\mathbf{P}_1$ et le second est un article publié dans le cadre du symposium Finite Volumes for Complex Applications V qui découle de mon travail de stage de Master concernant les conditions aux limites “ouvertes” pour les équations de Navier–Stokes incompressibles.

Chapitre 1

Synthèse générale

1.1 Introduction

L'Institut de Radioprotection et de Sûreté Nucléaire (IRSN) réalise des expertises et mène des recherches dans le cadre de la sûreté nucléaire et la protection de l'homme contre les rayonnements ionisants, du contrôle et de la protection des matières nucléaires. Dans le cadre de la problématique de sûreté des installations nucléaires, la Direction de Prévention contre les Accidents Majeurs (DPAM) mène un programme de recherche à la fois expérimental et en simulation numérique, concernant les incendies dans les milieux confinés et ventilés mécaniquement. C'est à cette fin qu'a été créé le code de calcul ISIS au sein du Laboratoire d'étude de l'Incendie et de Méthodes pour la Simulation et les Incertitudes (LIMSI), basé sur la plate-forme éléments finis C++ PELICANS également développée au sein du LIMSI. Les configurations d'écoulement qu'il traite principalement sont de type convection naturelle turbulente et se développent dans des espaces à l'échelle d'une ou plusieurs pièces d'une installation, tel que représenté sur la figure 1.1.

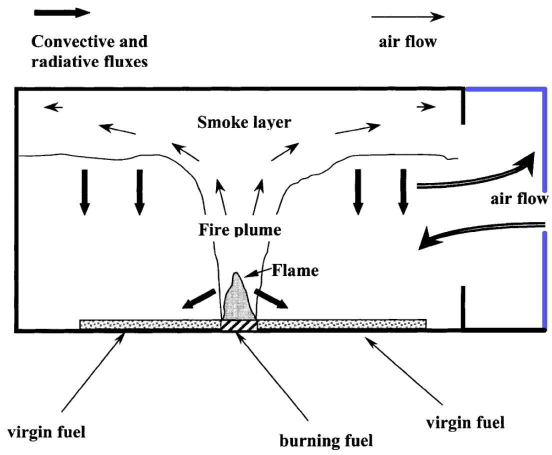

Figure 1.1 – Configuration modèle d'un incendie dans une installation

Les écoulements générés par un incendie dans un local confiné présentent ainsi des structures dont la taille couvre une gamme très large : des tourbillons de taille très petite (on parle de fluctuations turbulentes, qui peuvent ne mesurer que quelques microns dans les cas extrêmes) influent sur les mouvements d'ensemble à l'échelle du système (quelques mètres pour un local typique d'une installation nucléaire). Par ailleurs, ces écoulements sont essentiellement régis par les termes de flottabilité et peuvent présenter des zones de stratification, de relaminarisation ainsi que des recirculations. L'outil de simulation ISIS doit s'appuyer sur un modèle physique apte à en permettre une prédiction pertinente, tout en satisfaisant une contrainte d'efficacité en terme de temps de calcul.

Les modèles physiques de turbulence étudiés au cours de cette thèse sont dits “statistiques en un point” parce que les grandeurs physiques décrivant la turbulence sont évaluées en chaque point en fonction des corrélations des fluctuations temporelles des champs de vitesse et de pression. Ils ont été choisis car ils offrent le compromis recherché entre précision et coût de résolution. C'est plus particulièrement le cas des modèles en moyenne de Reynolds (RANS) de type $k-\varepsilon$ que nous avons choisi d'étudier. Ceux-ci se traduisent par deux équations additionnelles non-linéaires couplées aux équations de Navier–Stokes, décrivant le transport, pour l'une, de l'énergie cinétique turbulente et, pour l'autre, de son taux de dissipation. Ils résultent d'une analyse purement phénoménologique des transferts d'énergie aux différentes échelles de l'écoulement et s'appuient sur des hypothèses fortes telles que l'homogénéité et l'isotropie de la turbulence. La variante dite $k-\varepsilon$ RNG apporte une correction de la surestimation de l'énergie cinétique turbulente liée à cette dernière hypothèse, et permet ainsi une meilleure prédiction des écoulements en recirculation par rapport au modèle $k-\varepsilon$ standard. C'est le développement de schémas numériques pour ce dernier modèle qui a fait initialement l'objet du travail présenté dans le premier chapitre.

Néanmoins, on pourra remarquer que les modèles de turbulence RANS au premier ordre partagent tous la même structure quelles que soient les échelles turbulentes considérées, et que celles-ci étant dimensionnellement homogènes à une énergie, une fréquence de relaxation ou un taux de dissipation d'énergie, la même contrainte de positivité existe pour toutes ces échelles : les techniques utilisées dans le cadre des modèles présentés peuvent donc être également étendues à d'autres modèles RANS. Le modèle $k-\varepsilon-\overline{v^2}-f$, développé par Paul Durbin à partir d'arguments issus de la théorie des modèles de fermeture au second ordre (RSM) a ainsi également été étudié. Cette extension des modèles de type $k-\varepsilon$ adjoint une échelle turbulente supplémentaire $\overline{v^2}$ et une fonction f dite “elliptique” permettant de prendre notamment en compte l'anisotropie en zone proche parois et palier à “l'anomalie aux points d'arrêt” (*stagnation point anomaly*).

Du point de vue numérique, le système d'équations RANS complet est résolu par un schéma à pas fractionnaire : les équations de Navier-Stokes discrétisées par une technique d'éléments finis non-conformes, sont résolues par une méthode de projection, tandis que les équations de bilan du modèle de turbulence sont discrétisées par la méthode de volumes finis. Le travail effectué au cours de cette thèse a donné lieu à l'étude de trois problèmes permettant d'aborder les caractéristiques du schéma numérique.

D'une part, les schémas numériques proposés doivent respecter certaines contraintes imposées par les caractéristiques du modèle physique (notamment les bornes physiques des grandeurs turbulentes) et, par ailleurs, il est souhaitable qu'ils assurent également une stabilité inconditionnelle vis-à-vis du pas de temps. Dans le cas du système $k-\varepsilon$ dont les équations sont couplées, ces caractéristiques ne peuvent être obtenues que par la résolution par une méthode non-linéaire implicite. Une semi-discrétisation en temps adéquate de l'opérateur de convection et des termes sources des équations turbulentes a donc été proposée dans le cas des modèles de type $k-\varepsilon$, permettant de garantir la positivité des grandeurs turbulentes k et ε. Une brève étude sur l'établissement de loi de parois pour le modèle de turbulence $\overline{v^2}-f$ est également présentée : celle-ci a abouti à la réévalutation, vis-à-vis de la littérature, des constantes et des profils des grandeurs turbulentes adimensionées.

D'autre part, un résultat de convergence a été montré sur un problème réduit constitué des équations de Stokes incompressibles et d'une équation de convection-diffusion présentant un terme source de type production turbulente. Des résultats d'existence d'une solution pour des problèmes elliptiques couplés dans le cas continu, ainsi que des résultats de convergence pour des schémas de volumes finis et d'éléments finis de Lagrange, existent dans la littérature. Le résultat obtenu montre la convergence dans le cas d'un schéma utilisant l'élément fini de Crouzeix–Raviart pour l'approximation des équations de Stokes et la méthode de volumes finis pour l'équation de convection–diffusion. Une réflexion a été également menée sur le cas où les viscosités sont non-bornées et a pas abouti à montrer la stabilité du schéma.

Enfin, le second membre des équations de la turbulence présente la particularité de ne pas appartenir à L^2 comme cela est usuellement le cas dans les problèmes liés à la mécanique des fluides mais uniquement dans L^1. La convergence du schéma de volumes finis pour une équation de convection–diffusion instationnaire modèle des équations de bilan de la turbulence, a donc été étudiée. Cette étude présente l'originalité d'être un pas vers l'analyse du problème instationnaire et a nécessité l'établissement d'un résultat de compacité dans L^1 qui peut-être vu comme un équivalent discret du Lemme d'Aubin–Simon.

Le travail de thèse est synthétisé de la manière suivante. Dans un premier temps, les équations des modèles de turbulences qui ont fait l'objet d'une étude bibliographique sont décrites. Les modèle $k-\varepsilon$ standard, la variante $k-\varepsilon$ RNG et l'extension $k-\varepsilon-\overline{v^2}-f$ sont ici envisagés : les motivations du choix de ces modèles sont présentés ainsi que leurs caractéristiques du point de vue de la modélisation physique.

Dans une deuxième section, un problème mathématique général est formulé et les espaces d'approximation éléments finis et volumes finis sont introduits. Enfin l'algorithme complet de résolution du problème couplant les équations de Navier–Stokes et les modèle de turbulence est décrit : le schéma semi-discret en temps ainsi que le problème discret en espace associé au problème mathématique sont détaillés.

Enfin, les trois sous-problèmes d'analyse numérique décrits précédemment sont présentés et les résultats d'analyse obtenus pour chacun sont donnés ainsi que des éléments de preuve.

1.2 Équations des modèles physiques

Dans cette section, quelques éléments de modélisation de la turbulence en moyenne de Reynolds au premier ordre, issus d'une étude bibliographique approfondie, sont introduits dans un premier temps en prenant l'exemple du modèle $k-\varepsilon$ standard. Dans un deuxième temps, la variante RNG du modèle $k-\varepsilon$ qui permet une amélioration de la prédiction d'écoulement complexes, intéressante dans le cadres des écoulements rencontrés dans la modélisation de l'incendie, est décrite. Enfin le modèle $\overline{v^2}-f$, qui constitue une extension du modèle $k-\varepsilon$ issue de la modélisation au second ordre (RSM), est présenté. Dans chacun des exposés, les difficultés de modélisation physique, ayant mené à l'étude de ces modèles, sont abordées ainsi que les contraintes qui devront être prises en compte pour le développement des schémas numériques.

1.2.1 Introduction aux modèles de turbulence au premier ordre

On suppose qu'en tout point $(\boldsymbol{x}, t)$ de l'espace et du temps, l'écoulement est régi par les équations de bilan de quantité de mouvement et de masse. Il s'agit des équations de Navier–Stokes dites "instantanées", qui s'écrivent pour un fluide Newtonien en équilibre isotherme :

$$\partial_t(\rho \boldsymbol{u}) + (\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\cdot\big(2\mu\, \mathbf{D}(\boldsymbol{u})\big) - \boldsymbol{\nabla} p \tag{1.1a}$$

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}) = 0 \tag{1.1b}$$

où $\rho = \rho(\boldsymbol{x}, t)$ sera considéré ici comme une donnée (en pratique, et notamment dans le code ISIS, il est évalué à partir de variables intensives gouvernées par des équations de bilan ; par exemple,la température), et $\mathbf{D}(\boldsymbol{u})$ est le tenseur des taux de cisaillement, défini par $\mathbf{D}(\boldsymbol{u}) = 1/2(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^t\boldsymbol{u})$.

Les champs de vitesse $\boldsymbol{u}$ et de pression p instantanés sont décomposés en la somme d'un champ moyen et d'un champ fluctuant de moyenne nulle, étant donné un opérateur de moyenne " $\bar{\cdot}$ " vérifiant une hypothèse de commutativité avec les opérateur de dérivée. La décomposition (1.2) appliquée à tout champ scalaire ou toute composante d'un champ vectoriel est appelée "décomposition de Reynolds" et le système obtenu par l'application de l'opérateur de moyenne à chacun des termes des équations du système (1.1) est dénommé "équations de Navier–Stokes en moyenne de Reynolds" (RANS) [6].

$$q = q + q' \;,\; \overline{q'} = 0 \tag{1.2}$$

Si les opérateurs de moyenne et de dérivation commutent pour les termes linéaires, le terme de convection de l'équation de quantité de mouvement fait apparaître pour le champ moyen un terme lié aux corrélations de vitesses $-\boldsymbol{\nabla}\cdot(\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'})$. Ce terme peut s'interpréter comme la divergence d'un tenseur modélisant les contraintes additionnelles dues aux processus turbulents et qui est appelé tenseur de Reynolds.

Le problème de "fermeture" consiste à modéliser le tenseur de Reynolds $-\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}$ comme une fonction des quantités moyennes. Dans le cadre des modèles $k - \varepsilon$ linéaires, on suppose que l'action des contraintes turbulentes se traduit par une diffusion additionnelle pour l'écoulement moyen. Cette hypothèse appelée "hypothèse de Boussinesq" mène à la définition du déviateur, noté $\mathbf{R}$, du tenseur des contraintes de Reynolds :

$$\mathbf{R} = \mathrm{Dev}(-\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}\,) = 2\mu_t \mathbf{D}(\bar{\boldsymbol{u}}) - \frac{2}{3}(\rho k + \mu_t \boldsymbol{\nabla}\cdot\boldsymbol{u})\,\mathbf{I} \tag{1.3}$$

où $\mathbf{D}(\bar{\boldsymbol{u}}) = 1/2(\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{\nabla}^t\bar{\boldsymbol{u}})$ est le tenseur de cisaillement moyen et μ_t est un coefficient de proportionnalité strictement positif que l'on nomme viscosité turbulente. La prise en compte des effets turbulents conduit donc à remplacer, dans l'équation de quantité de mouvement, la viscosité intrinsèque du fluide μ_ℓ, dite également viscosité laminaire, par une viscosité effective μ telle que $\mu = \mu_\ell + \mu_t$. Le second terme $2/3\rho k\,\mathbf{I}$ représente la partie sphérique du tenseur de Reynolds et apparaît dans l'équation comme un terme de pression supplémentaire, appelé "pression cinétique turbulente", et usuellement non-explicité.

Dans le cas des modèles au premier ordre, la viscosité turbulente μ_t est déterminée grâce à sune relation algébrique faisant intervenir des échelles scalaires caractéristiques de la turbulence. Dans la relation suivante, connue sous le nom de l'hypothèse de Prandtl–Kolmogorov, interviennent l'énergie cinétique turbulente k et son taux de dissipation ε :

$$\mu_t = \rho\, C_\mu \, \frac{k^2}{\varepsilon} \tag{1.4}$$

où $k = 1/2\overline{|\boldsymbol{u}'|^2}$, $\varepsilon = 2\mu/\rho\overline{|\mathbf{D}(\boldsymbol{u}')|^2}$ et C_μ est un coefficient constant.

Les grandeurs turbulentes k et ε sont évaluées grâce aux deux équations de bilan scalaires établies à partir des équations de Navier–Stokes pour les fluctuations :

$$\partial_t(\rho k) + \boldsymbol{\nabla}\cdot(\rho k \bar{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\boldsymbol{\nabla} k\right) = \mathbf{R} : \boldsymbol{\nabla}\bar{\boldsymbol{u}} - \rho\varepsilon \tag{1.5a}$$

$$\partial_t(\rho\varepsilon) + \boldsymbol{\nabla}\cdot(\rho\varepsilon\bar{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\boldsymbol{\nabla}\varepsilon\right) = \frac{\varepsilon}{k}\left(C_{\varepsilon 1}\mathbf{R} : \boldsymbol{\nabla}\bar{\boldsymbol{u}} - \rho\, C_{\varepsilon 2}\,\varepsilon\right) \tag{1.5b}$$

Ces deux équations sont couplées par les termes sources et les coefficients de diffusion, tous deux étant des fonctions non-linéaires des inconnues. Les coefficients $C_{\varepsilon 1}$ et $C_{\varepsilon 2}$ sont généralement constants et évalués de manière empirique, ainsi que les nombres de Prandtl–Schmidt σ_k et σ_ε issus d'une hypothèse de gradient moyen.

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε
0.09	1.44	1.92	1.0	1.3

Table 1.1 – Constantes du modèle $k - \varepsilon$ standard

1.2.2 Modèle $k - \epsilon$ RNG *"Groupe de Renormalisation"*

Le modèle $k - \varepsilon$ standard étant peu performant dans des configurations d'écoulement présentant des recirculations, il a semblé intéressant d'évaluer les performances d'une de ses variantes, le modèle $k-\varepsilon$ RNG (ReNormalization Group). La première version du modèle RNG a été proposée par Yakhot et Orszag [30], puis une série d'adaptations [19, 28, 20, 31] a abouti au modèle révisé [32]. Ces modèles résultent d'une approche novatrice qui applique les techniques de renormalisation, développées à l'origine pour la théorie des champs en physique quantique et la théorie cinétique des gaz, à l'établissement des équations de bilan pour k et ε.

L'emploi de ces méthodes mathématiques repose sur certaines hypothèses fortes mais permet de réévaluer de manière rigoureuse les constantes empiriques du modèle $k - \varepsilon$ standard : les valeurs obtenues sont regroupées dans la table 1.2.

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε
0.0837	1.42	1.68	0.7194	0.7194

Table 1.2 – Constantes du modèle $k - \varepsilon$ RNG

Il apparaît également un terme source $\mathbf{S}_{\mathrm{RNG}}$ supplémentaire dans l'équation de bilan de la variable ε, qui permet une correction locale du taux de dissipation de l'énergie. Les équations du modèle, tel qu'utilisé dans ISIS, sont données dans la table 1.3 ; elles font apparaître un terme source supplémentaire, noté $\mathbf{G}$, qui résulte d'une modélisation empirique des phénomènes de génération ou de destruction de la turbulence induits par les forces de flottabilité, cruciaux dans les écoulements en convection naturelle.

L'expérience montre que l'utilisation du modèle $k - \varepsilon$ RNG permet de corriger la surestimation de l'énergie cinétique turbulente par le modèle $k - \varepsilon$ standard dans des zones de forte déformation plane, phénomène désigné dans la littérature comme "l'anomalie aux points d'arrêt" [9]. Des travaux ultérieurs ont toutefois permis de montrer que le bon comportement du modèle ne doit pas être attribué à un établissement plus rigoureux des équations, mais plutôt à une compensation d'erreur entre les termes de production et de dissipation turbulente. La surestimation de l'énergie cinétique aux points d'arrêt est en effet une conséquence de l'hypothèse d'isotropie sur laquelle se basent tous les modèles de type $k-\varepsilon$. Il n'en reste pas moins que le modèle RNG est actuellement le modèle de référence utilisé le plus souvent dans les applications industrielles.

Équations de bilan :

$$\partial_t(\rho k) + \boldsymbol{\nabla}\cdot(\rho k \bar{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\boldsymbol{\nabla} k\right) = \mathbf{P} + \mathbf{G} - \rho\varepsilon$$

$$\partial_t(\rho \varepsilon) + \boldsymbol{\nabla}\cdot(\rho \varepsilon \bar{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\boldsymbol{\nabla} \varepsilon\right) = \frac{\varepsilon}{k}\left(C_{\varepsilon 1}\,\mathbf{P} + C_{\varepsilon 1ga}\mathbf{G} + C_{\varepsilon 1gb}\mathbf{G}^{+} - \rho\, C_{\varepsilon 2}\,\varepsilon\right) + \mathbf{S}_{\mathrm{RNG}}$$

avec :

$$\mathbf{P} = 2\mu_t|\mathbf{D}(\bar{\boldsymbol{u}})|^2 - \frac{2}{3}\rho k \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}$$

$$\mathbf{S}_{\mathrm{RNG}} = -\rho C_\mu C_\eta \eta^2 \frac{\varepsilon^2}{k} + \rho C_{\varepsilon 3}\,\varepsilon\,\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}$$

$$\mathbf{G} = \mu_t \frac{\boldsymbol{\nabla}\rho\cdot\boldsymbol{g}}{\sigma_g \rho} \qquad \text{où } \boldsymbol{g} \text{ désigne la gravité ,} \quad \mathbf{G}^{+} = \max(0,\mathbf{G})$$

$$C_\eta = \tfrac{\eta(1-\eta/\eta_0)}{1+\beta\eta^3} \qquad \text{où} \quad \eta_0 = 4.38, \quad \eta = \sqrt{2|\mathbf{D}(\bar{\boldsymbol{u}})|^2}\; k/\varepsilon, \quad \beta = 0.012$$

$$C_{\varepsilon 3} = \frac{1}{3}\left[-1 + 2C_{\varepsilon 1} - 3m_1(n_1 - 1) + (-1)^{\delta}\sqrt{6}C_\mu C_\eta \eta\right]$$

$$\text{où } m_1 = 0.5\ ,\ n_1 = 1.4,\ \delta = 1 \text{ si } \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} < 0,\ \delta = 0 \text{ si } \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} > 0$$

Table 1.3 – Équations du modèle $k-\varepsilon$ RNG

1.2.3 Modèle $k - \varepsilon - \overline{v^2} - f$

Dans le cadre des modèles de type $k-\varepsilon$, les variables k et ε permettent d'évaluer l'intensité turbulente et le temps de relaxation caractéristique de la turbulence de manière adéquate sous l'hypothèse de turbulence homogène isotrope. Néanmoins, ceux-ci présentent généralement des faiblesses dans le cas de configurations d'écoulements complexes présentant des recirculations, notamment l'anomalie aux points d'arrêt. Si, comme on l'a vu, des corrections comme celle de $k-\varepsilon$ RNG permettent d'améliorer la prédiction des écoulements, cette amélioration ne peut contourner le défaut principal de cette classe de modèle : l'utilisation d'une grandeur isotrope dans les zones proche parois où l'anisotropie joue un rôle essentiel.

L'établissement du modèle $\overline{v^2} - f$ est inspiré de la modélisation au second ordre et mène à l'ajout d'une échelle de vitesse au modèle $k-\varepsilon$ standard afin d'améliorer le traitement de la turbulence en zone proche paroi. Cette faiblesse du modèle standard est intrinsèque car elle découle du choix d'une grandeur isotrope telle que k en tant qu'échelle caractéristique de vitesse : en zone proche paroi, ce choix est inadéquat et mène à une évaluation erronée de la viscosité turbulente. Les zones de paroi s'assimilant à des zones de déformation plane, l'échelle de vitesse $\overline{v^2}$, mettant en jeu la fluctuation de vitesse normale aux lignes de courant semble plus appropriée et permet ainsi la prise en compte de l'*effet de blocage cinématique*.

Ce choix permet d'assurer la décroissance de la viscosité turbulente sans avoir recours à une fonction d'amortissement [29]. Il permet par ailleurs de s'affranchir de l'utilisation d'une loi de paroi logarithmique dont la validité n'est pas assurée à bas nombre de Reynolds. On introduit par ailleurs une fonction f de répartition de l'énergie cinétique turbulente, évaluée par une équation dite de *relaxation elliptique* [10] qui permet notamment de traduire l'effet non-local de la paroi, appelé *écho de paroi*, dû à la diffusion turbulente par les fluctuations de pression. Cette fonction est définie d'après des arguments de la modélisation de la turbulence au second ordre, et s'interprète comme la somme des tensions de dissipation et des corrélations de pression.

La viscosité turbulente est ainsi évaluée grâce à une relation algébrique similaire à (1.4) et faisant intervenir $\overline{v^2}$:

$$\mu_t = \rho C_\mu \overline{v^2} \mathrm{T} \tag{1.6}$$

où T représente le temps de retournement k/ε borné inférieurement par l'échelle temporelle de Kolmogorov.

Les équations du modèle $k-\varepsilon-\overline{v^2}-f$ sont regroupées dans la table 1.5 et les constantes du modèles dans la table 1.4.

Les équations de bilan des variables k et ε sont similaires au modèle $k-\varepsilon$ standard, à l'exception du coefficient $C_{\varepsilon 1}$ qui est modélisé de manière à prendre en compte l'anisotropie locale. Dans le modèle original [8], la constante $C^z_{\varepsilon 1}$ est dépendante de d, la distance à la plus proche paroi :

$$C^z_{\varepsilon 1} = 1.3 + \frac{0.25}{1+(d/2l)^8}$$

La valeur $C^z_{\varepsilon 1} = 1.3$ évaluée expérimentalement correspond à un écoulement libre et est recouvrée par l'expression choisie dans la limite $d \to \infty$, tandis que la valeur $C^z_{\varepsilon 1} = 1.55$ évaluée également expérimentalement est obtenue en proche paroi. Dans la pratique l'évaluation de d pose problème et d'autres expressions algébriques de $C^z_{\varepsilon 1}$ ont été envisagées [25, 22].

L'équation de bilan de $\overline{v^2}$ possède une structure similaire à celle de k. Elle présente un terme de production $\rho k f$ qui prend en compte la répartition de l'énergie cinétique turbulente par dissipation et par l'effet d'*écho de parois*, et un terme de destruction piloté par le temps de retournement $\mathrm{T} \sim k/\varepsilon$. L'équation elliptique pour f présente un second membre similaire à celui de l'équation de bilan de k à un facteur $1/\rho k$ près par homogénéité, et dont les termes de production et destruction sont pondérés par les constantes C_1 et C_2. L'échelle $\lambda \sim k^3/\varepsilon^2$ contrôle la longueur caractéristique de diffusion de l'énergie cinétique turbulente en zone proche parois et est minorée par l'échelle spatiale de Kolmogorov.

La reformulation du traitement des parois grâce au système $\overline{v^2}-f$ implique la réévaluation des constantes du modèle $k-\varepsilon$ ainsi que l'introduction de nouvelles constantes.

La constante C_μ voit sa valeur fortement réduite dans le modèle standard, ce qui est expliqué par le fait qu'elle se comporte comme le rapport d'anisotropie $\overline{u'v'}^2/k^2$, où u' et v' représentent respectivement les composantes tangentielle et normale de la fluctuation de vitesse.

Les valeurs des constantes introduites par le modèle $\overline{v^2} - f$ ont été déterminées par des calculs de simulation numérique directe.

C_μ	$C^z_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε	C_1	C_2	C_L	C_{T}	C_η
0.19	$C_{\varepsilon 1} + \max\left(1.55 - C_{\varepsilon 1}, 0.045\sqrt{\frac{k}{\overline{v^2}}}\right)$	1.9	1.3	1.0	1.4	0.3	0.3	6.0	70.0

Table 1.4 – Constantes du modèle $k - \varepsilon - \overline{v^2} - f$

Équations de bilan :

$$\partial_t(\rho k) + \boldsymbol{\nabla}\cdot(\rho k \bar{\mathbf{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_k}\right)\boldsymbol{\nabla} k\right) = \mathbf{P} - \rho\varepsilon$$

$$\partial_t(\rho \varepsilon) + \boldsymbol{\nabla}\cdot(\rho \varepsilon \bar{\mathbf{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_\varepsilon}\right)\boldsymbol{\nabla} \varepsilon\right) = \frac{1}{T}\left(C^z_{\varepsilon 1}(\overline{v^2}, k)\mathbf{P} - \rho C_{\varepsilon 2}\varepsilon\right)$$

$$\partial_t(\rho \overline{v^2}) + \boldsymbol{\nabla}\cdot(\rho \overline{v^2} \bar{\mathbf{u}}) - \boldsymbol{\nabla}\cdot\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_k}\right)\boldsymbol{\nabla} \overline{v^2}\right) = \rho k f - \rho\frac{\overline{v^2}}{T}$$

$$f - \lambda^2 \Delta f = C_2 \frac{\mathbf{P}}{\rho k} - (1 - C_1)\frac{2/3 - \overline{v^2}/k}{\mathrm{T}}$$

tel que :

$$\mathbf{P} = 2\mu_t |\mathbf{D}(\bar{\mathbf{u}})|^2 - \frac{2}{3}\rho k \boldsymbol{\nabla}\cdot\bar{\mathbf{u}}$$

$$\mathrm{T} = \max\left(\frac{k}{\varepsilon},\ C_T\,\tau_K\right)$$

$$\lambda = C_L l \qquad \text{avec} \qquad l^2 = \max\left(\frac{k^3}{\varepsilon^2}, C_\eta^2\,(\eta_K)^2\right)$$

Échelles spatiales et temporelles de Kolmogorov :

$$\tau_K = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}} \qquad \text{et} \qquad \eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}$$

Table 1.5 – Équations du modèle $k - \varepsilon - \overline{v^2} - f$

1.3 Modèle mathématique et discrétisations

Dans cette section, après avoir décrit de manière générale le problème mathématique correspondant aux modèles physiques de turbulence au premier ordre, on introduit les discrétisations volumes finis et éléments finis non-conformes qui sont utilisées dans les schémas numériques étudiés pour la résolution des modèles de turbulence en moyenne de Reynolds. On présente ensuite, dans un premier temps, l'algorithme complet semi-discrétisé en temps pour une itération, puis les équations discrétisées en espace correspondantes.

1.3.1 Problème

On considère le problème mathématique suivant, issu de la modélisation de la turbulence au premier ordre à P équations, défini sur un domaine ouvert borné, connexe Ω de $\mathbb{R}^d$, $d = 2, 3$ et sur un intervalle de temps fini $(0, T)$ partitionné de manière uniforme, avec un pas (constant) $\delta t = t^{n+1} - t^n$, $0 \leq n < N$. Il est constitué des équations de Navier–Stokes instationnaires pour un fluide Newtonien gouvernant l'évolution des champs de vitesse et de pression moyens (qui seront désormais notés $\boldsymbol{u}$ et p par souci de concision dans le cadre de l'analyse mathématique), ainsi que d'un ensemble d'équations de convection–diffusion de champs scalaires $\{\chi_i\}_{1\leq i\leq \mathrm{P}}$ représentant les échelles turbulentes.

On cherche $(\boldsymbol{u}, p, \{\chi_i\})$ vérifiant $\forall(\boldsymbol{x}, t) \in \Omega \times (0, T)$:

$$\partial_t(\rho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \boldsymbol{\nabla}\cdot\Big(\mu\big(\{\chi_j\}\big)\boldsymbol{\nabla}\boldsymbol{u}\Big) + \boldsymbol{\nabla}p = \mathbf{g} \tag{1.7a}$$

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0 \tag{1.7b}$$

$$\partial_t(\rho\chi_i) + \boldsymbol{\nabla}\cdot(\rho\chi_i\,\boldsymbol{u}) - \boldsymbol{\nabla}\cdot\Big(\lambda_{\chi_i}\big(\{\chi_j\}\big)\nabla\chi_i\Big) = f_{\chi_i}(\{\chi_j\}) \tag{1.7c}$$

$$\chi_i(\boldsymbol{x}, t) > 0 \tag{1.7d}$$

$$\boldsymbol{u}(\boldsymbol{x}) = 0\ ,\ \ \chi_i(\boldsymbol{x}) = \chi_i|_{\partial\Omega} \geq 0 \qquad \forall\boldsymbol{x}\in\partial\Omega \tag{1.7e}$$

avec $\mathbf{g}$ un terme de forçage, ρ la masse volumique du fluide, μ la viscosité effective et λ_{χ_i} le coefficient de diffusion de l'équation de bilan de χ_i. On suppose par ailleurs qu'il existe un nombre réel $\mu_\ell > 0$ représentant la viscosité intrinsèque du fluide tel que $\mu \geq \mu_\ell$. On admet également, sans perte de généralité, que $\lambda_{\chi_i} \approx \mu$, ce qui correspond dans le modèle physique à nombre de Prandtl–Schmidt turbulent constant.

Conformément à l'hypothèse de Boussinesq (1.3), on considère par ailleurs que le second membre des équations de bilan turbulentes (1.7c) peut s'écrire sous la forme générale suivante :

$$f_{\chi_i}(\{\chi_j\}) = \alpha_i(\{\chi_j\})\lambda_{\chi_i}\big(\{\chi_j\}\big)|\boldsymbol{\nabla}\boldsymbol{u}|^2 - \chi_i\beta_i(\{\chi_j\})\ ,\ \ 1 \leq i, j \leq \mathrm{P} \tag{1.8}$$

tel que α_i et β_i sont deux fonctions continues pour $\chi_j \in (0, \infty)$, strictement positives et bornées, possiblement non-linéaires vis-à-vis des variables χ_j.

Quelques remarques peuvent être formulées concernant le problème 1.7 :

1. L'expression du terme de second membre (1.8) est cohérente avec la structure de production turbulente / dissipation obtenue par l'établissement des équations de la turbulence par décomposition en moyenne de Reynolds. Elle est non-linéaire vis-à-vis des variables χ_i et de plus, de nombreux termes correctifs permettant d'améliorer les performances des modèles dans des configurations d'écoulement spécifiques peuvent lui être ajoutés. La condition (1.7d) de positivité n'étant donc *a priori* pas assurée, nous supposons qu'elle est vérifiée par les modèles physiques utilisés : la construction d'un schéma numérique garantissant la positivité pour un système de deux équations de la turbulence a donc été abordé (Section 1.4).

2. Par ailleurs, les équations (1.7a) et (1.7c) sont couplées par les coefficients et par le terme de production turbulente $\lambda_{\chi_i}(\{\chi_j\})|\boldsymbol{\nabla u}|^2$ présent au second membre de (1.7c). Le couplage entre les équations de Navier–Stokes et le modèle de turbulence joue un rôle central dans la preuve d'existence d'une solution au problème continu et doit être envisagé pour l'analyse de convergence du schéma numérique. D'une part, il s'agit de s'assurer que la discrétisation du terme de production turbulente est adéquate : on a donc vérifié la convergence du schéma numérique choisi, sur le problème modèle de Stokes stationnaire muni d'une équation convection–diffusion scalaire (Section 1.5). D'autre part, les coefficients de diffusion μ, λ_{χ_i} sont généralement des fonctions non-bornées des variables $\{\chi_j\}$: quelques réflexions ont donc été menées pour tenter de montrer la convergence du schéma numérique sur le même problème modèle dans le cas où λ est non-bornée.

3. Enfin, le second membre $\boldsymbol{g}$ est supposé appartenir à $\mathrm{L}^2(\Omega \times (0,T))^d$, ce qui nous place dans le cadre fonctionnel usuel de l'analyse des équations de Navier–Stokes. En supposant que les coefficients de diffusion $\mu, \{\lambda_i\}$ sont bornés, les estimations d'énergie classiques impliquent que $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)^d) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^d)$: le second membre f_{χ_i} défini par l'expression (1.8) appartient donc au mieux à $\mathrm{L}^1(0,T;L^1(\Omega))$. Il est donc légitime de se poser la question de la convergence du schéma de volumes finis pour une équation de convection–diffusion instationnaire à données L^1 telle que (1.7c) (Section 1.6).

1.3.2 Espaces discrets

Soit $\mathcal{M}$ une partition du domaine Ω en quadrilatères ($d = 2$) ou en hexahèdres convexes ($d = 3$) ou en simplexes, tel que $\bar{\Omega} = \cup_{K\in\mathcal{M}}\bar{K}$. Pour tout élément $K \in \mathcal{M}$, $\partial K = \bar{K} \setminus K$ représente la frontière de K. La famille $\mathcal{E}$ représente l'ensemble des ouverts bornés non-vides de $\mathbb{R}^{d-1}$ contenus dans $\bar{\Omega}$ tels qu'il existe un hyperplan E de $\mathbb{R}^d$ vérifiant $\sigma = \partial K \cap E$: dans le cas $d = 2$ il s'agit des arêtes des éléments et dans le cas $d = 3$, de leurs faces. Ainsi pour tout élément $K \in \mathcal{M}$, il existe un sous-ensemble noté $\mathcal{E}(K) \subset \mathcal{E}$ de sorte que $\mathcal{E} = \cup_{K\in\mathcal{M}}\mathcal{E}(K)$. On note $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$) l'ensemble des arêtes internes (resp. externes) défini par $\mathcal{E}_{\rm int} = \big\{\sigma : \sigma \cap$
bound $= \{0\}\big\}$ (resp. $\mathcal{E}_{\rm ext} = \big\{\sigma : \sigma \cap \partial\Omega \neq \{0\}\big\}$). Pour chaque face interne $\sigma = K|L$ du maillage, $\boldsymbol{n}_{\sigma,K}$ désigne le vecteur normal à σ, orienté de K vers L. On désignera de manière indistincte par $|\cdot|$ la mesure de Lebesgue en dimension d ou $d-1$ (par exemple, $|K|$ désigne la mesure d-dimensionnelle d'un élément, et $|\partial K|$ la mesure (d-1)-dimensionnelle de sa frontière).

La partition $\mathcal{M}$ est supposée régulière dans le sens usuel éléments finis, notamment elle vérifie les propriétés suivantes :

Definition 1.3.1 (Maillage régulier)**.**

1. Pour tout couple d'éléments $(K,L) \in \mathcal{M}^2$ alors $\bar{K} \cap \bar{L}$ est soit réduit à $\{0\}$, à un point ou, dans le cas $d = 3$, à un segment, soit $\bar{K} \cap \bar{L}$ est l'arête (ou face) commune à K et L notée $\sigma = K|L$.
2. La régularité du maillage est caractérisée par le paramètre $\theta_{\mathcal{M}} > 0$ tel que :

$$\theta_{\mathcal{M}} = \inf\left\{\frac{\xi_K}{h_K}; K \in \mathcal{M}\right\} \tag{1.9}$$

   où ξ_K et h_K, repésentent respectivement le diamètre du cercle inscrit dans K et le diamètre de K.

Dans l'algorithme considéré, la discrétisation spatiale des équations de Navier–Stokes repose sur une technique d'éléments finis non-conformes de bas degré, tandis que les équations de convection–diffusion pour les échelles turbulentes sont discrétisées par une méthode de volumes finis standard. L'élément fini de Crouzeix–Raviart est utilisé dans le cas de maillages simplexes [7] et l'élément fini de Rannacher–Turek [27] dans le cas de maillages quadrilatéraux ou hexaédriques. Ainsi, dans les discrétisations mixtes choisies pour l'approximation des équations de Navier–Stokes, les degrés de liberté de vitesse sont situés au centre des faces et la pression au centre des volumes de contrôles. Ces discrétisations satisfont une condition de stabilité de type Ladhyzenskaia–Babuska–Brezzi. Par ailleurs, elle sont particulièrement adaptées au couplage des équations de Navier–Stokes avec des équations volumes finis de convection–diffusion scalaires, car elles permettent de satisfaire l'équation de bilan de masse et ainsi d'assurer la monotonicité de l'opérateur de convection volumes finis.

L'élément de référence $\hat{K}$ pour l'élément de Crouzeix–Raviart est le d-simplexe unitaire et l'espace discret est l'espace des fonctions affines $\mathrm{P}_1(\hat{K})^d = \mathrm{span}\{1, (x_i)_{1\leq i\leq d}\}$, tandis que l'élément de référence pour l'élément de Rannacher–Turek est d–cube et l'espace discret est :

$$\tilde{\mathrm{Q}}_1(\hat{K})^d = \mathrm{span}\{1, (x_i)_{1\leq i\leq d}, (x_i^2 - x_{i+1}^2)_{1\leq i<d}\}$$

La transport de l'élément de référence sur le maillage est la transformation affine pour l'élément de Crouzeix–Raviart et la transformation Q_1 pour l'élément de Rannacher–Turek. Les degrés de liberté sont déterminés par les fonctionnelles suivantes :

$$\forall \sigma \in \mathcal{E}(K), \qquad \mathrm{F}_\sigma(v) = \frac{1}{|\sigma|}\int_\sigma v \,\mathrm{d}\gamma \tag{1.10}$$

où γ représente la mesure de Lebesgue $(d-1)$-dimensionnelle.

L'espace discret de la vitesse noté $\boldsymbol{V}_h$ est défini par :

$$\begin{aligned}\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \mathrm{L}^2(\Omega)^d \ : \ & v_h|_K \in V(K)^d, \forall K \in \mathcal{M}\,;\\ & v_{\sigma,i} \text{ continu à travers chaque face } \sigma \in \mathcal{E}_{\mathrm{int}},\ 1\leq i\leq d\,;\\ & v_{\sigma,i} = 0,\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}},\ 1\leq i\leq d\,;\}\end{aligned}$$

Pour toute fonction $\boldsymbol{v}_h \in \boldsymbol{V}_h$, l'ensemble des degrés de liberté correspondant est déterminé par la relation (1.10) pour toute composante v_i :

$$\{v_{\sigma,i} = \mathrm{F}_\sigma(v_i) :\ \forall \sigma \in \mathcal{E}_{\mathrm{int}},\ 1\leq i\leq d\}$$

On définit $v_\sigma = \sum_{1\leq i\leq d} v_{\sigma,i}\boldsymbol{e}_i$, $\boldsymbol{e}_i$ étant le i-ème vecteur de la base canonique de $\mathbb{R}^d$ et on note $\boldsymbol{\phi}_\sigma^{(i)} = \phi_\sigma \boldsymbol{e}_i$ la fonction associée à $v_{\sigma,i}$, telle que la fonction de forme scalaire ϕ_σ satisfait de manière usuelle pour tout couple de faces $\xi, \sigma \in \mathcal{E}$:

$$\mathrm{F}_\xi(\phi_\sigma) = \begin{cases} 1 & \text{si } \xi = \sigma \\ 0 & \text{sinon} \end{cases}$$

On définit l'opérateur d'interpolation r_h suivant, conformément au cadre théorique décrit dans l'article originel de Crouzeix–Raviart [7] dans le cas général des éléments finis mixtes non-conformes pour le problème de Stokes :

$$\left|\begin{array}{lcll} r_h & : & \mathrm{H}_0^1(\Omega) & \to \ V_h \\ v & & & \mapsto \ v_h = r_h v = \displaystyle\sum_{\sigma\in\mathcal{E}} v_\sigma \phi_\sigma = \sum_{\sigma\in\mathcal{E}} \left(\frac{1}{|\sigma|}\int_\sigma v(\boldsymbol{x})\,\mathrm{d}\gamma\right)\phi_\sigma \end{array}\right. \tag{1.11}$$

L'opérateur d'interpolation de $\mathrm{H}_0^1(\Omega)^d$ dans $\boldsymbol{V}_h$ peut être naturellement construit en appliquant r_h à chaque composant des fonctions de $\mathrm{H}_0^1(\Omega)^d$ et sera noté $\boldsymbol{r}_h$ dans la suite.

La pression est approchée par l'espace discret $\mathrm{H}_\mathcal{M}$ des fonctions constantes par élément de $\mathcal{M}$:

$$\mathrm{H}_\mathcal{M} = \{q_h \in \mathrm{L}^2(\Omega) : q_h|_K = \text{ constant}, \forall K \in \mathcal{M}\}$$

et l'ensemble des degrés de liberté pour la pression est défini par :

$$\left\{p_K = \frac{1}{K}\int_K p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} : \forall K \in \mathcal{M}\right\}$$

L'opérateur $\boldsymbol{r}_h$ vérifie, pour toute fonction $v \in \mathrm{H}_0^1(\Omega)^d$, une propriété de conservation faible de la divergence :

$$\forall K \in \mathcal{M}, \qquad \int_K q_h \boldsymbol{\nabla}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_K q_h \boldsymbol{\nabla}\cdot\boldsymbol{r}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \qquad \forall q_h \in \mathrm{H}_\mathcal{M}$$

Étant donné que seule la continuité faible de la solution est assurée à travers les faces du maillage, les vitesses peuvent être discontinues aux faces : la discrétisation est ainsi non-conforme dans $\mathrm{H}_0^1(\Omega)^d$. Pour toute fonction v_h appartenant à l'espace discret V_h, on peut définir un gradient discret $\boldsymbol{\nabla}_h$ tel que

$\boldsymbol{\nabla}_h v_h = \{\boldsymbol{\nabla}_{h,i} v_h\}_{1\leq i\leq d}$ où $\boldsymbol{\nabla}_{h,i} v_h$ est une fonction constante par morceaux et appartenant à $\mathrm{L}^2(\Omega)^d$ qui est égale à la dérivée de v_h presque partout. Cet opérateur discret peut être étendu à toute fonction à valeur vectorielle $\boldsymbol{v}_h \in \boldsymbol{V}_h$ et l'opérateur de divergence discret correspondant sera noté $\boldsymbol{\nabla}_h \cdot$.

L'espace V_h est muni de la semi-norme de Sobolev H^1–brisée :

$$\forall v \in V_h \ , \qquad \|v\|_{1,b} = \left(\int_\Omega |\boldsymbol{\nabla}_h v|^2 \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}} = \left(\sum_{K\in\mathcal{M}} \|\boldsymbol{\nabla} v\|^2_{\mathrm{L}^2(K)}\right)^{\frac{1}{2}}$$

qui est également une norme pour V_h étant donné que des conditions de Dirichlet homogènes sont appliquées à la frontière.

Afin de pouvoir construire une approximation volumes finis consistante du Laplacien on suppose qu'il existe une famille de points $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ telle que $x_K \in K$ pour $K \in \mathcal{M}$ et telle que pour tout face interne $\sigma = K|L$, la droite passant par x_K et x_L soit orthogonale à σ. Pour tout volume de contrôle K et toute face $\sigma \in \mathcal{E}(K)$, $d_{K,\sigma}$ représente la distance euclidienne entre x_K et σ.

De plus, pour l'approximation du terme de convection on définit la quantité

$$\mathrm{v}_{K,\sigma} = \int_{\sigma=K|L} \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n}_{\sigma,K} \,\mathrm{d}\gamma$$

et pour définir l'approximation *upwind* on note respectivement $\mathrm{v}^+_{\sigma,K}$ et $\mathrm{v}^-_{\sigma,K}$ les quantités $\mathrm{v}^+_{\sigma,K} = \max(0, \mathrm{v}_{K,\sigma})$ et $\mathrm{v}^-_{\sigma,K} = -\min(0, \mathrm{v}_{K,\sigma})$. Pour toute arête ou face σ, on note $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, si σ sépare les volumes de contrôle K et L (auquel cas d_σ est la distance euclidienne entre x_K et x_L), sinon $d_\sigma = d_{K,\sigma}$ si σ est inclus dans $\partial\Omega$.

On note $\mathrm{H}_\mathcal{M}(\Omega)$ l'espace des fonctions constantes par morceaux sur chaque volume de contrôle de $\mathcal{M}$. L'opérateur d'interpolation naturel volumes finis est défini par :

$$\left| \begin{array}{lllll} \pi_\mathcal{M} & : & \mathrm{L}^1(\Omega) & \to & \mathrm{H}_\mathcal{M}(\Omega) \\ & & v & \mapsto & v_\mathcal{M} = \displaystyle\sum_{K\in\mathcal{M}} v_K 1_K \end{array} \right. \tag{1.12}$$

avec $v_K = \dfrac{1}{|K|}\displaystyle\int_K v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, pour tout volume de contrôle $K \in \mathcal{M}$.

L'espace d'approximation $\mathrm{H}_\mathcal{M}(\Omega)$ est muni du produit scalaire

$$\big[u, v\big]_\mathcal{D} = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\sigma=K|L} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(v_K - v_L) + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}\, u_K v_K$$

et on définit la forme bilinéaire

$$\big\langle u, v\big\rangle_\mathcal{D} = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\sigma=K|L} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(v_K - v_L)$$

correspondant respectivement aux discrétisations volumes finis de l'opérateur de Laplace muni de conditions aux limites de Dirichlet homogènes et de Neumann homogènes. On définit les norme et semi-norme suivantes :

$$\forall u \in \mathrm{H}_\mathcal{M}(\Omega)\ , \quad \|u\|_{1,\mathcal{M}} = \big[u, u\big]^{\frac{1}{2}}_\mathcal{D} \quad \text{and} \quad |u|_{1,\mathcal{M}} = \big\langle u, u\big\rangle^{\frac{1}{2}}_\mathcal{D}$$

De manière similaire pour toute fonction $u \in \mathrm{H}_\mathcal{M}(\Omega)$ on définit la norme de Sobolev $\mathrm{W}^{1,q}(\Omega)$ discrète, pour $q \in [1, \infty)$, par :

$$\|u\|_{1,q,\mathcal{M}} = \left(\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\sigma=K|L} |\sigma|\ d_\sigma \left|\frac{u_K - u_L}{d_\sigma}\right|^q + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}(K)} |\sigma|\ d_\sigma \left|\frac{u_K}{d_\sigma}\right|^q\right)^{\frac{1}{q}}$$

1.3.3 Algorithme de résolution du problème

Nous décrivons maintenant l'algorithme de résolution du problème modèle (1.7) composé des équations de Navier–Stokes et d'une ou plusieurs équations de bilan pour une variable scalaire positive caractéristique de la turbulence.

1.3.3.1 Schéma semi-discret

Les équations de Navier–Stokes et du modèle de turbulence au premier ordre sont résolues grâce à un schéma à pas fractionnaire décrit ici dans un formalisme semi-discret à tout instant t^{n+1}, $1 \leq n < N$:

1. Trouver $\{\chi_i^{n+1}\}_{1\leq i\leq \mathrm{P}}$ tel que :

$$\frac{\rho^n \chi_i^{n+1} - \rho^{n-1}\chi_i^n}{\delta t} + \boldsymbol{\nabla}\cdot(\chi_i^{n+1}\rho^n \boldsymbol{u}^n) - \boldsymbol{\nabla}\cdot\Big(\lambda_{\chi_i}(\{\chi_j^n\})\nabla\chi_i^{n+1}\Big) \\ = \alpha_i(\{\chi_j^n\}) - \chi_i^{n+1}\beta_i(\{\chi_j^{n+1},\chi_j^n\}) \quad (1.13)$$

   avec $1 \leq j \leq \mathrm{P}$, α_i et β_i deux fonctions vérifiant $\alpha_i > 0$ et $\beta_i \geq 0$

2. Trouver $(\tilde{\boldsymbol{u}}^{n+1}, p^{n+1})$ tel que :

$$\frac{\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1}\boldsymbol{u}^n}{\delta t} + \boldsymbol{\nabla}\cdot(\rho^n \tilde{\boldsymbol{u}}^{n+1}\otimes \boldsymbol{u}^n) - \boldsymbol{\nabla}\cdot\Big(\mu(\{\chi_j^n\})\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^{n+1}\Big) + \boldsymbol{\nabla}p^n = \boldsymbol{f}^{n+1} \quad (1.14a)$$

$$\rho^n \frac{\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla}(p^{n+1} - p^n) = 0 \quad (1.14b)$$

   Conservation de la masse à l'instant t^{n+1} :

$$\frac{\rho^{n+1} - \rho^n}{\delta t} + \boldsymbol{\nabla}\cdot(\rho^{n+1}\boldsymbol{u}^{n+1}) = 0 \quad (1.14c)$$

La première étape de l'algorithme consiste en la résolution des équations du système turbulent de manière couplée. Ce choix est dicté par le couplage des termes sources des équations de la turbulence et leur nature non-linéaire. Une stratégie de semi-discrétisation des termes sources, basée sur un argument algébrique de M-matrice, est utilisée afin préserver la positivité de la solution $\{\chi_i^{\chi}\}_{1\leq i\leq \mathrm{P}}$ quel que soit le pas de temps pourvu que l'équation (1.14c) soit vérifiée au pas de temps précédent. Le terme de diffusion est discrétisé de manière linéairement implicite, c'est-à-dire que les coefficients de diffusion λ_{χ_i} sont évalués en fonction des champs χ_i calculés au pas de temps précédent.

La deuxième étape est la résolution des équations de Navier–Stokes (1.14) par une méthode de projection incrémentale. L'équation de bilan de quantité de mouvement (1.14a) est discrétisée de manière semi-implicite : la pression ainsi que les flux convectifs sont évalués au pas de temps précédent. Elle est tout d'abord résolue afin de prédire un champ de vitesse $\tilde{\boldsymbol{u}}^{n+1}$ qui ne satisfait pas le bilan de masse (1.14c).

Enfin, le champ prédit est projeté grâce à la résolution d'une équation de Darcy (1.14b) qui peut être reformulée en un problème de Laplace pour la pression. La troisième étape est cruciale car elle permet de satisfaire l'équation de conservation de la masse. La stabilité de l'algorithme complet repose en effet sur un résultat de stabilité l'opérateur de convection volumes finis détaillé en (1.6.3). Ce résultat permet de tirer parti de la monotonicité de l'opérateur d'advection au niveau discret pourvu que le bilan de masse soit satisfait. La conservation de la masse étant assurée en fin d'itération en temps, les masses volumiques sont ainsi décalées en temps dans le terme de convection. Par ailleurs, il est crucial que l'approximation des flux convectifs $(\rho\boldsymbol{u})^n$ dans ces deux équations soit identique à celle de (1.14c) : les éléments finis non-conformes de bas degré utilisés, dont les inconnues de vitesse sont situées au barycentre des faces, sont ainsi parfaitement adaptés au couplage avec une méthode de volumes finis.

1.3.3.2 Problème discret

Pour tout volume de contrôle $K \in \mathcal{M}$, on note ρ_K une approximation de la masse volumique ρ sur K et on suppose que la famille des nombres réels $(\rho_K)_{K\in\mathcal{M}}$ est strictement positive. Par souci de clarté, nous utilisons une notation relative à un pas de temps : pour toute fonction scalaire ou vectorielle v les inconnues à l'instant t^{n+1} ne sont pas indicées en temps, tandis que les champs évalués respectivement aux instants t^n et t^{n-1} sont notés v^* et v^{**}. Le problème discret associé au problème 1.7 avec une échelle turbulente χ s'écrit :

$$\begin{aligned}\forall K \in \mathcal{M}, \qquad & \frac{|K|}{\delta t}\big(\rho_K^*\chi_K - \rho_K^{**}\chi_K^*\big) + \sum_{\sigma=K|L}\big((F_{\sigma,K}^*)^+\chi_K - (F_{\sigma,K}^*)^-\chi_L\big) \\ & + \sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{|\sigma|}{d_\sigma}\lambda_\sigma\big(\{\chi_j^*\}\big)(\chi_K-\chi_L) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}}\frac{|\sigma|}{d_{K,\sigma}}\lambda_\sigma\big(\{\chi_j^*\}\big)(\chi_K-\chi_\sigma) \\ & = |K|\Big[\alpha\big(\{\chi_j^*\}\big) - \beta\big(\{\chi_j,\chi_j^*\}\big)\chi_K\Big] \qquad (1.15a)\end{aligned}$$

avec $F_{\sigma,K}^* = |\sigma|(\rho\boldsymbol{u})_\sigma^*$ le flux massique évalué au pas de temps précédent, sortant de K à travers la face σ.

La discrétisation du bilan de quantité de mouvement est détaillée dans l'Annexe B, et notamment la discrétisation de l'opérateur de convection B.2.2.

De plus, on donne l'équation discrète de conservation de la masse qui sera utilisé dans les sections suivantes :

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K - \rho_K^*) + \sum_{\sigma\in\mathcal{E}(K)} F_{\sigma,K} = 0 \qquad (1.16)$$

1.4 Schéma numérique monotone pour les modèles de turbulence à deux équations

Comme il l'a été introduit dans la section 1.3.1, il est crucial du point de vue de la physique que la positivité des inconnues du modèle de turbulence soit assurée. La difficulté principale consiste à satisfaire cette contrainte dans le cadre de la résolution d'un système d'équations de convection–diffusion couplées, et ceci de préférence indépendamment du pas de temps choisi. Si la littérature sur les performances des modèles $k-\varepsilon$ est substantielle, la stabilité des schémas implicites pour les modèles de turbulence à deux équations est une question peu abordée : on peut citer, par exemple, un schéma préservant la positivité de la solution pour le modèle $k-\varepsilon$ dans le cadre de la résolution par la méthode des caractéristiques [23], et un schéma de différences finies appliqué à une variante du modèle $k-\omega$ incompressible utilisant la propriété de M-matrice [24]. Dans cette première étude, on envisage l'extension de cette dernière technique aux modèles de turbulence de type $k-\varepsilon$ dans le cas à masse volumique variable. Le problème est constitué de deux équations de type $k-\varepsilon$ couplées, où l'on suppose le champ de vitesse $\boldsymbol{u}$ connu et suffisamment régulier :

$$\partial_t(\rho k) + \boldsymbol{\nabla}\cdot(\rho k\boldsymbol{u}) - \boldsymbol{\nabla}\cdot\Big(\mu_k(k,\varepsilon)\boldsymbol{\nabla}k\Big) = \mathbf{P} - \rho\varepsilon$$

$$\partial_t(\rho\varepsilon) + \boldsymbol{\nabla}\cdot(\rho\varepsilon\boldsymbol{u}) - \boldsymbol{\nabla}\cdot\Big(\mu_\varepsilon(k,\varepsilon)\boldsymbol{\nabla}\varepsilon\Big) = \frac{\varepsilon}{k}\Big(C_{\varepsilon1}\mathbf{P} - \rho C_{\varepsilon2}\varepsilon\Big)$$

où on rappelle que μ_k et μ_ε sont les viscosité effectives associées aux équations de k et ε définies telles que $\mu_\chi = \mu_\ell + \mu_t/\sigma_\chi$ avec $\mu_\ell, \sigma_\chi > 0$ pour $\chi = \{k,\varepsilon\}$, $\mathbf{P} \geq 0$ représente la production turbulente et $C_{\varepsilon1}$, $C_{\varepsilon2}$ sont deux nombres réels strictement positifs dépendant du modèle physique considéré. De plus le problème est muni de conditions initiales strictement positives :

$$k(\boldsymbol{x}, t=0) > 0, \ \varepsilon(\boldsymbol{x}, t=0) > 0 \text{ pour presque tout } \boldsymbol{x} \in \Omega \qquad (1.17)$$

Soit $\{t^n\}_{0\leq n\leq N}$, $N \in \mathbb{N}$, une partition uniforme de l'intervalle $[0,T]$ sur lequel on cherche une solution, le schéma numérique Euler implicite choisi s'écrit pour tout instant t^n, $0 \leq n < N$:

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t}\big(\rho_K^n k_K^{n+1} - \rho_K^{n-1} k_K^n\big) + \sum_{\sigma=K|L} F_{\sigma,K}^n k_\sigma^{n+1}$$
$$+ \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{|\sigma|}{d_\sigma}\mu_k(k^n,\varepsilon^n)_\sigma\big(k_K^{n+1} - k_L^{n+1}\big) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \frac{|\sigma|}{d_{K,\sigma}}\mu_k(k^n,\varepsilon^n)_\sigma k_K^{n+1} = |K|\mathbf{S_k}^{n+1}(k^{n+1},k^n,\varepsilon^{n+1},\varepsilon^n) \tag{1.18a}$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t}\big(\rho_K^n \varepsilon_K^{n+1} - \rho_K^{n-1} \varepsilon_K^n\big) + \sum_{\sigma=K|L} F_{\sigma,K}^n \varepsilon_\sigma^{n+1}$$
$$+ \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{|\sigma|}{d_\sigma}\mu_\varepsilon(k^n,\varepsilon^n)_\sigma\big(\varepsilon_K^{n+1} - \varepsilon_L^{n+1}\big) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \frac{|\sigma|}{d_{K,\sigma}}\mu_\varepsilon(k^n,\varepsilon^n)_\sigma \varepsilon_K^{n+1} = |K|\mathbf{S}_\varepsilon{}^{n+1}(k^{n+1},k^n,\varepsilon^{n+1},\varepsilon^n) \tag{1.18b}$$

où k_σ, ε_σ sont discrétisés de manière décentrée amont, et tel que $F_{\sigma,K}^n = |\sigma|(\rho^n \boldsymbol{u}^n)_\sigma$ le flux massique sortant de K à travers la face σ et évalué au pas de temps précédent, ρ_K^n et ρ_K^{n-1} deux nombres réels positifs qui sont supposés vérifier $\forall K \in \mathcal{M}$:

$$\frac{|K|}{\delta t}\big(\rho_K^n - \rho_K^{n-1}\big) + \sum_{\sigma=K|L} F_{\sigma,K} = 0 \tag{1.19}$$

La conservation de la positivité de la solution à tout pas de temps repose sur un résultat algébrique utilisant la propriété de M-matrice suivante :

Lemme 1.4.1. *Soit A une matrice de $\mathbb{R}^{M\times M}$, $M \in \mathbb{N}$ telle que, pour tout $1 \leq i \leq M$, les hypothèses suivantes sont vérifiées :*

1. $A_{i,i} > 0$
2. $A_{i,j} \leq 0, \ \forall\ 1 \leq j \leq M, \ j \neq i$
3. $\displaystyle\sum_{1\leq j\leq M} A_{i,j} > 0$

alors A est une matrice non-singulière appelée M-matrice, et elle admet une matrice inverse A^{-1} positive, c'est-à-dire que $A_{ij}^{-1} \geq 0$, $\forall i,j$.

Soient $X^{n+1} = (k^{n+1}, \varepsilon^{n+1})^t$ et $X^n = (k^n, \varepsilon^n)^t$ les vecteurs associés aux champs discrets inconnus k et ε aux instants t^{n+1} et t^n. Supposons que l'on puisse écrire le système 1.18 sous la forme $A(X^n, X^{n+1})\, X^{n+1} = f(X^n, X^{n+1})$ où $A(X^n, X^{n+1})$ et $f(X^n, X^{n+1})$ sont une matrice et un second membre respectivement, dépendant éventuellement des inconnues (donc le système n'est pas nécessairement linéaire par rapport à X^{n+1}), mais dont on peut garantir les propriétés suivantes : A est une M-matrice et toutes les composantes de f sont positives. Alors, pour tout $0 \leq n \leq N$, toutes les composantes de X^{n+1} sont positives, ce qui implique la positivité de k et ε.

Une discrétisation adéquate de chacun des termes des équations doit être utilisée de sorte que la matrice du système linéaire vérifie les hypothèses du Lemme 1.4.1 Le traitement de l'opérateur d'advection $s \mapsto \partial_t(\rho s) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u} s)$ se base notamment sur un résultat de monotonicité de l'opérateur discret issu de [21] (qui étudie le cas d'un écoulement compressible et avec une discrétisation volumes finis décentrée amont) ; on y démontre que l'opérateur discret associé vérifie un principe du maximum sous la condition qu'il s'annule lorsque l'inconnue est constante, ce qui, au vu de son expression, peut être interprété comme le fait que le bilan de masse $\partial_{t,\mathcal{D}}\rho + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho\boldsymbol{u})$ discret est satisfait. Dans un algorithme à pas fractionnaire tel qu'implémenté dans ISIS, cette condition peut être satisfaite par un décalage en temps des masses volumiques de manière similaire à l'argument développé dans [18]. En effet, l'équation de bilan de masse (1.19) étant satisfaite au pas de temps précédent par les inconnues discrètes, la somme des termes de l'opérateur de convection, assemblés dans la matrice du système linéaire s'écrit :

$$\forall K \in \mathcal{M}, \quad \sum_{K' \in \mathcal{M}} A_{K,K'} = \sum_{L \in \mathcal{N}(K)} A_{K,L} = |K| \frac{\rho_K}{\delta t} + \sum_{\sigma = K|L} F_{\sigma,K} = |K| \frac{\rho_K^*}{\delta t} > 0 \tag{1.20}$$

Sous réserve d'expliciter les coefficients de diffusion, la même propriété est vérifiée par la discrétisation en volumes finis usuelle de l'opérateur de diffusion.

Le traitement des termes sources est effectué de la manière suivante. Considérons dans le cas général un terme source de la forme $f(k, \varepsilon)$. Deux cas se présentent : soit $f(\cdot)$ peut être rendue positive par une semi-discrétisation en temps *ad hoc* en utilisant le fait que k et ε sont supposés positifs au pas de temps précédent, soit $f(\cdot)$ est négative. Dans le premier cas, ce terme source est laissé au second membre. Entrent dans cette catégorie les fonctions du type $f(\varepsilon) = \varepsilon$ que l'on discrétisera comme ε^n ou $f(\varepsilon) = \varepsilon^2$, pour laquelle on a le choix entre $(\varepsilon^{n+1})^2$ ou $(\varepsilon^n)^2$. Dans le second cas, on écrit le terme source sous la forme $\varepsilon^{n+1} f/\varepsilon^n$, et ce terme est inclus au membre de gauche, ce qui renforce la diagonale de la quantité positive $-f/\varepsilon^n$. Par exemple, à tout instant t^n la discrétisation d'un terme $\mathbf{S} = \mathbf{G}$ au second membre de l'équation de bilan pour la variable k (1.18a) s'écrit :

$$\mathbf{S}^n = \mathbf{G}^n \quad \text{si } \mathbf{G}^n \geq 0 \quad \text{et} \quad \mathbf{S}^n = \mathbf{G}^n \frac{k^{n+1}}{\max(k^n, k^*)} \quad \text{sinon}$$

où k^* est un paramètre destiné à limiter le terme de production pour les très faibles valeurs de k^n.

On peut alors proposer par exemple la semi-discrétisation en temps des termes sources suivante dans le cas du modèle $k - \varepsilon$ RNG incompressible :

$$\left| \begin{array}{l} \mathbf{S_k}^{n+1} = \mathbf{P}^n - \rho^n |\varepsilon^{n+1}| \dfrac{k^{n+1}}{\max(k^n, k^*)} \\ \mathbf{S}_\varepsilon{}^{n+1} = \gamma^n (C_{\varepsilon 1} \mathbf{P}^n - \rho^n (C_{\varepsilon 2} + C_{\text{rng}} \text{sgn}(C_{\varepsilon r}^+)) \dfrac{(\varepsilon^{n+1})^2}{\max(k^n, k^*)} - \rho^n C_{\text{rng}} \text{sgn}(C_{\varepsilon r}^-) \dfrac{(\varepsilon^n)^2}{\max(k^n, k^*)} \end{array} \right. \tag{1.21}$$

avec $\gamma^n = \dfrac{\rho^{n-1} C_\mu k^n}{\mu_t^n}$ soit $\gamma^n \sim \dfrac{\varepsilon^n}{k^n}$, $C_{\text{rng}} = C_\mu C_\eta \eta^2$, $C_{\varepsilon r} = C_{\varepsilon 2} + C_{\text{rng}}$ et k^* fixé. De plus $x^+ = |x|$ et $x^- = |-x|$.

Du point de vue théorique, on montre que le schéma numérique satisfait le théorème suivant :

Théorème 1.4.1. *Supposant que les valeurs initiales de k et ε sont strictement positives, il existe une unique solution au schéma numérique* (1.18)*, et par ailleurs cette solution est strictement positive.*

Positivité : La conservation de la positivité de la solution à tout pas de temps se base sur les arguments développés ci-dessus.

Existence : La preuve d'existence se base sur un argument de degré topologique. En utilisant la stratégie de discrétisation des termes sources précédemment décrite, les équations de volumes finis obtenues sont de type convection–diffusion–réaction avec un coefficient de réaction et un second membre strictement positifs, or celles-ci vérifient un principe du maximum. On peut donc montrer que, pour un pas de temps donné, la solution vérifie une estimation L^∞ dépendant des données et de la solution au pas de temps précédent.

On construit ensuite une fonction continue $F(k, \varepsilon, \xi)$ de telle sorte que les termes source non-linéaires sont multipliés par un paramètre $\xi \in [0, 1]$ et l'on considère l'équation $F(k, \varepsilon, \xi) = 0$: pour $\xi = 1$ l'équation obtenue correspond au problème (non-linéaire) considéré et pour $\xi = 0$ l'équation dégénère en une équation de convection–diffusion–réaction linéaire. Par ailleurs, on vérifie que l'estimation L^∞ de la solution, initialement montrée, est indépendante de ξ ; le lemme de degré topologique nécessitant une estimation de la solution uniforme en ξ.

Le degré topologique de F étant non-nul pour $\xi = 0$ et invariant par homotopie, on déduit donc que le problème correspondant à $F(k, \varepsilon, 1) = 0$ admet une solution.

Unicité : on suppose qu'il existe deux solutions $X_1 = (k_1, \varepsilon_1)^t$ et $X_2 = (k_2, \varepsilon_2)^t$ au problème discret (1.18), et en soustrayant terme à terme on se ramène à une équation sur $\delta X = X_1 - X_2$. En utilisant le résultat de principe du maximum pour l'équation de bilan de $\delta\varepsilon$ (qui est decouplée de celle de k), on montre que $\delta\varepsilon = 0$, puis en utilisant $\varepsilon_1 = \varepsilon_2$ dans l'équation de bilan de δk on obtient $\delta k = 0$, ce qui conclut la preuve.

1.5 Analyse de convergence d'un schéma éléments finis/volumes finis pour un problème modèle

Dans l'algorithme de résolution décrit dans la section 1.3.3, on résout successivement les équations d'un modèle de turbulence à deux équations tel que celui traité dans la section précédente et les équations de Navier-Stokes. Alors que les équations de convection–diffusion de scalaires sont discrétisées par la méthode de volumes finis usuelle, les équations de Navier-Stokes sont résolues grâce à une technique d'éléments finis non-conformes. Ces éléments finis permettent d'assurer une approximation consistante de la divergence du champ de vitesse par maille : cette propriété permet, d'une part, d'assurer la monotonicité de l'opérateur de convection volumes finis pour les équations de bilan des échelles turbulentes, et d'autre part, il permet d'obtenir une discrétisation cohérente du terme de production turbulente avec celle du terme de diffusion de l'équation de bilan de quantité de mouvement, ce qui a une importance cruciale pour assurer la convergence du schéma numérique. Par ailleurs, afin de se conformer au modèle mathématique il semble intéressant d'étudier la convergence de l'algorithme complet en affaiblissant l'hypothèse sur la viscosité turbulente, et de considérer à la fois les cas d'un problème à viscosité turbulente bornée et non-bornée.

On se propose dans cette partie de montrer, sur un problème modèle, la stabilité du schéma numérique et un résultat de convergence, dans le cas d'un viscosité bornée, au sens où, considérant une suite $\big(\mathcal{M}^{(m)}\big)_{m\in\mathbb{N}}$ de discrétisations telle que $h_{\mathcal{M}} \to 0$ quand $m \to \infty$, la suite des solutions discrètes converge (éventuellement à une sous-suite près) vers une solution du problème continu (en un sens à définir). Dans le cas des viscosités non-bornées, on ne peut malheureusement pas conclure à la convergence du schéma. On considère le problème modèle suivant sur un ouvert borné connexe Ω de $\mathbb{R}^d$, $d = 2, 3$:

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\big(\lambda(k)\boldsymbol{\nabla}\boldsymbol{u}\big) + \boldsymbol{\nabla} p &= \boldsymbol{f} \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 \end{aligned} \tag{1.22a}$$

$$-\boldsymbol{\nabla}\cdot\big(\lambda(k)\boldsymbol{\nabla} k\big) + \boldsymbol{\nabla}\cdot\big(k\boldsymbol{u}\big) = \lambda(k)|\boldsymbol{\nabla}\boldsymbol{u}|^2 \tag{1.22b}$$

$$\boldsymbol{u}(\boldsymbol{x}) = 0\ ,\ \ k(\boldsymbol{x}) = 0 \qquad \forall \boldsymbol{x} \in \partial\Omega \tag{1.22c}$$

avec $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$.

Ce problème modèle permet d'aborder certaines difficultés que présente l'analyse du système d'équations proposé en 1.22. Les équations de Stokes et l'équation de bilan de k sont en effet couplées, à la fois, par la viscosité turbulente dépendant de k, présente dans le terme de diffusion de (1.22a), et par le terme de production turbulente $\lambda(k)|\boldsymbol{\nabla}\boldsymbol{u}|^2$ au second membre de l'équation (1.22b). Ces termes possèdent des propriétés particulières qui imposent de se placer dans un cadre fonctionnel différent de celui rencontré classiquement dans l'analyse des équations de la mécanique des fluides.

D'une part, si l'on considère la forme faible de l'équation (1.22a) :

$$\int_\Omega \lambda(k)\boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \tag{1.23}$$

en prenant $\boldsymbol{v} = \boldsymbol{u}$, il vient naturellement qu'il existe une constante $C > 0$ dépendant de f et μ telle que $\|\sqrt{\lambda(k)}\boldsymbol{\nabla}\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)^{d\times d}} \le C$, sous l'hypothèse que λ est bornée inférieurement par un nombre réel strictement positif. Le second membre de l'équation (1.22b) n'est donc pas d'énergie finie mais seulement borné dans $\mathrm{L}^1(\Omega)$.

L'analyse des équations elliptiques et paraboliques non-linéaires avec second membre irrégulier a été largement traitée dans les travaux de Boccardo–Gallouët [2] et dans une série d'articles ultérieurs [17, 1, 3] dans le cas continu. Dans ce cadre, l'existence et l'unicité de la solution d'une classe de problème elliptiques non-linéaires a été montrée pour $d = 2$, ainsi que dans le cas $d = 3$ sous certaines hypothèses de type condition d'entropie. Les estimations *a priori* en normes discrètes correspondantes ont été montrées dans le cas du Laplacien volumes finis par Gallouët et Herbin [13] ainsi que la convergence du schéma numérique. Ces résultats ont été étendus au cas d'un problème de dissipation par effet Joule, constitué de deux équations elliptiques couplées avec viscosités bornées, par Bradji et Herbin [5] pour des discrétisations volumes finis et éléments finis de Lagrange. L'existence d'une solution à un problème constitué de deux équations de diffusion couplées a été montrée par Gallouët, Lederer, Lewandowski et Tartar [15], dans le cas de viscosités non-bornées.

Dans ce dernier cas, les hypothèses générale suivantes, (utilisées dans le [15, Théorème 2.1], peuvent être envisagées :

Hypothèses 1.5.1 (Contrôle de la viscosité turbulente)**.** On suppose $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction non-bornée :

1. il existe un nombre réel positif $\mu > 0$ tel que pour tout $k \in \mathbb{R}^+$, $\lambda(k) > \mu$,
2. il existe trois nombres réels $C_1 \geq 0$, $C_2 > 0$, $\gamma > 1/2$ tels que :

$$\forall\, k \in [0,1] \qquad \lambda'(k) \leq C_1$$
$$\forall\, k \in [1,+\infty) \qquad \frac{\left(\sqrt{\lambda(k)}\right)'}{\sqrt{\lambda(k)}} \leq \frac{C_2}{k^\gamma}$$

Par exemple, ces hypothèses sont satisfaites par une viscosité, de type *viscosité de mélange*), $\lambda(k) = \sqrt{\mu^2 + \ell^2 k}$, avec ℓ et μ deux nombres réels respectivement positif et strictement positif ; ainsi pour tout $k \in \mathbb{R}^+$, $\lambda(k) \geq \mu > 0$.

Pour l'analyse de convergence, nous serons amenés à supposer que la viscosité est bornée. Une telle viscosité peut être obtenue, par exemple, en tronquant la viscosité précédente :

$$\lambda(k) = \max(\sqrt{\mu^2 + \ell^2 k}, \bar{\lambda}) \tag{1.24}$$

pour tout $k \in \mathbb{R}^+_\star$, avec $\bar{\lambda}$ un réel positif.

Le problème discret envisagé, constitué des équations de Stokes stationnaires incompressibles discrétisées par une méthode d'éléments finis non-conformes de Crouzeix–Raviart et d'une équation de convection–diffusion stationnaire approchée par la méthode volumes finis standard, s'écrit :

$$\forall \sigma \in \mathcal{E}_{\rm int}\,,\ 1 \leq i \leq d, \qquad \int_\Omega \lambda(k) \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \phi^i_\sigma \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \boldsymbol{\nabla}_h \cdot \phi^i_\sigma \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \phi^i_\sigma \,\mathrm{d}\boldsymbol{x} \tag{1.25a}$$

$$\forall q \in \mathrm{H}_\mathcal{M} \quad \int_\Omega q \boldsymbol{\nabla}_h \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = 0 \tag{1.25b}$$

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma = K|L} \frac{|\sigma|}{d_\sigma} \lambda(k)_\sigma \big(k_K - k_L\big) + \sum_{\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)} \frac{|\sigma|}{d_{K,\sigma}} \lambda(k)_\sigma \big(k_K - k_\sigma\big)$$
$$+ \sum_{\sigma = K|L} \big(\mathrm{v}^+_{\sigma,K} k_K - \mathrm{v}^-_{\sigma,K} k_L\big) = |K| \Big[\lambda(k) |\boldsymbol{\nabla}_h \boldsymbol{u}|^2\Big]_K \tag{1.25c}$$

Sous les hypothèses décrites, on montre que le schéma numérique choisi vérifie la propriété suivante dans le cas d'une viscosité bornée :

Théorème 1.5.1 (Résultat de convergence). *Soit $\big(\mathcal{M}^{(m)}\big)_{m \in \mathbb{N}}$ une suite de discrétisations du domaine Ω telle que $h^{(m)} \to 0$ quand $m \to \infty$. De plus, on suppose que toute discrétisation est admissible au sens où il existe un paramètre de régularité de maillage $\theta_0 > 0$ tel que $\theta^{(m)}_\mathcal{M} \geq \theta_0$, $\forall m \in \mathbb{N}$ avec $\theta^{(m)}_\mathcal{M}$ défini par la relation* (1.9)*.*

Pour tout $m \in \mathbb{N}$, on note respectivement $\boldsymbol{V}_h{}^{(m)}$ et $\mathrm{H}_{\mathcal{M}}^{(m)}$, les espace discrets de la vitesse et de la pression ou l'énergie cinétique turbulente, associés à $\mathcal{M}^{(m)}$, et par $\left(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)}\right) \in \boldsymbol{V}_h{}^{(m)} \times \mathrm{H}_{\mathcal{M}}^{(m)} \times \mathrm{H}_{\mathcal{M}}^{(m)}$ une solution du problème discret (1.25)*. Alors, sous l'hypothèse de viscosité 1.24 et pour toute suite de fonctions $\left(\boldsymbol{f}^{(m)}\right)_{m\in\mathbb{N}}$ telle que $\boldsymbol{f}^{(m)} \to \boldsymbol{f}$ dans $\mathrm{L}^2(\Omega)^d$, les résultats suivants sont vérifiés :*

1. *$\left(\boldsymbol{u}^{(m)}\right)_{m\in\mathbb{N}}$ converge fortement dans $\mathrm{L}^2(\Omega)^d$ vers $\bar{\boldsymbol{u}} \in \mathrm{H}_0^1(\Omega)^d$,*
2. *$\left(p^{(m)}\right)_{m\in\mathbb{N}}$ converge faiblement dans $\mathrm{L}^\alpha(\Omega)$, for $1 \le \alpha < 2$ vers $\bar{p} \in \mathrm{L}^\alpha(\Omega)$,*
3. *$\left(k^{(m)}\right)_{m\in\mathbb{N}}$ fortement dans $\mathrm{L}^s(\Omega)$, pour $1 \le s < q^*$ vers $\bar{k} \in \cup_{1\le q<d/(d-1)} \mathrm{W}_0^{1,q}(\Omega)$, avec $q^* = dq/(d-q)$.*

et $(\bar{\boldsymbol{u}}, \bar{p}, \bar{k})$ est une solution de (1.22) *au sens où elle satisfait le problème variationnel suivant :*

$$\left|\begin{array}{c} \textit{Trouver } (\boldsymbol{u}, p, k) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^\alpha(\Omega) \times \mathrm{W}_0^{1,q}(\Omega) \\ \textit{tel que pour tout } (\boldsymbol{v}, q, \psi) \in \mathrm{C}_c^\infty(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{C}_c^\infty(\Omega) \ : \\[2ex] \displaystyle\int_\Omega \lambda(k) \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \, \boldsymbol{\nabla}\cdot\boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \\ \displaystyle\int_\Omega q \boldsymbol{\nabla}\cdot\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = 0 \\ \displaystyle\int_\Omega \lambda(k) \boldsymbol{\nabla} k \cdot \boldsymbol{\nabla} \psi \, \mathrm{d}\boldsymbol{x} - \int_\Omega k \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \, \mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k) |\boldsymbol{\nabla} \boldsymbol{u}|^2 \psi \, \mathrm{d}\boldsymbol{x} \end{array}\right. \tag{1.26}$$

Estimations a priori et existence : Dans un premier temps, on montre les estimations *a priori* du lemme suivant puis on prouve l'existence d'une solution par un argument de point fixe.

Lemme 1.5.2. *Soit $\mathcal{M}$ une discrétisation admissible du domaine Ω et $\theta_0 > 0$ un nombre réel tel que $\theta_{\mathcal{M}}^{(m)} \ge \theta_0$, avec $\theta_{\mathcal{M}}^{(m)}$ défini par la relation* (1.9)*. Alors, si l'hypothèse 1.5.1 est satisfaite, il existe une solution $(\boldsymbol{u}, p, k) \in \boldsymbol{V}_h \times \mathrm{H}_{\mathcal{M}} \times \mathrm{H}_{\mathcal{M}}$ au problème discret 1.25 et les estimations suivantes sont vérifiées :*

1. *$\|\boldsymbol{u}\|_{1,b} \le C_1(\Omega, \boldsymbol{f}, \mu)$*
2. *$\|\sqrt{\lambda} \boldsymbol{\nabla}_h \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)^{d\times d}} \le C_2(\Omega, \boldsymbol{f}, \mu)$*
3. *$\|k\|_{1,q,\mathcal{D}} \le C_3(\Omega, \boldsymbol{f}, \mu, \phi)$ pour $1 \le q < d/(d-1)$*
4. *$\|k\|_{\mathrm{L}^s(\Omega)} \le C_4(\Omega, \boldsymbol{f}, \mu, \phi)$ pour $1 \le s < d/(d-2)$*
5. *$\|p\|_{\mathrm{L}^\alpha(\Omega)} \le C_5(\Omega, \boldsymbol{f}, \mu)$ pour $1 \le \alpha < 2\beta/(\beta+2)$ si $\sqrt{\lambda(k_{\mathcal{M}})} \in \mathrm{L}^\beta(\Omega)$ avec $\beta \in [2, \infty)$ dans le cas d'une viscosité non-bornée, et $\alpha = 2$ si elle est bornée.*

Compacité : selon les résultat classiques (rappelés dans [14] par exemple), la famille de solutions approchées $\left\{\boldsymbol{u}^{(m)}\right\}_{m\in\mathbb{N}}$ est compacte dans $\mathrm{L}^2(\Omega)$ et par ailleurs on montre que le gradient discret converge faiblement dans $\mathrm{L}^2(\Omega)$, il existe donc une fonction $\bar{\boldsymbol{u}} \in \mathrm{H}_0^1(\Omega)$ telle que $\boldsymbol{u}^{(m)} \to \bar{\boldsymbol{u}}$ fortement dans $\mathrm{L}^2(\Omega)$. La pression étant bornée dans $\mathrm{L}^\alpha(\Omega)$ indépendamment de m, il existe $\bar{p}$ tel que $p^{(m)} \to \bar{p}$ faiblement dans $\mathrm{L}^\alpha(\Omega)$. Enfin grâce au théorème de Kolmogorov, en utilisant le résultat détaillé dans [5], on montre qu'il existe $\bar{k} \in \mathrm{W}_0^{1,q}(\Omega)$ tel que $k^{(m)} \to \bar{k}$ fortement dans $\mathrm{L}^s(\Omega)$, $s < dq/(d-q)$.

Passage à la limite : on montre enfin que la limite $(\bar{\boldsymbol{u}}, \bar{p}, \bar{k})$ est solution du problème en passant à la limite dans chaque terme des équations. Le passage à la limite dans les équations de Stokes est classique.

On montre que $\sqrt{\lambda(k^{(m)})} \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}$ converge fortement dans $\mathrm{L}^2(\Omega)$ en prouvant la convergence faible $\sqrt{\lambda(k^{(m)})} \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}$ dans $\mathrm{L}^2(\Omega)$ ainsi que la convergence de la norme de Sobolev brisée vers la norme

$W^{1,2}(\lambda;\Omega)$. Le premier point découle directement des résultats de compacité obtenus tandis que la preuve de convergence des normes est remarquable car elle nécessite d'utiliser l'équation de quantité de mouvement. Tout d'abord en combinant le passage dans les équations de Stokes et le résultat de densité, on montre que $\boldsymbol{u}$ satisfait l'égalité :

$$\int_\Omega \lambda|\boldsymbol{\nabla}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_\Omega p\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \tag{1.27}$$

Puis en remarquant que pour tout $\boldsymbol{v}\in W_0^{1,2}(\lambda;\Omega)$:

$$\int_\Omega \lambda(k_{\mathcal{M}}^{(m)})\boldsymbol{\nabla}_h\boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h\boldsymbol{r}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}^{(m)}\cdot\boldsymbol{r}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega p^{(m)}\boldsymbol{\nabla}_h\cdot\boldsymbol{r}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

on peut choisir $\boldsymbol{v}=\boldsymbol{u}$ et passer à la limite dans le second membre de l'équation en utilisant les résultats de compacité : ce qui montre la convergence du second membre de l'équation volumes finis.

Le passage à la limite dans les autres termes de l'équation de k se base essentiellement, pour le terme de diffusion, sur un lemme de convergence faible du gradient volumes finis discrétisé sur le maillage diamant dans $L^s(\Omega)$, et, pour le terme de convection sur une preuve issue de [11] utilisé dans le cas de l'équation de bilan de masse, où les hypothèses de régularité sur ρ sont plus faible que celles de k.

Remarque 1.5.3 (Cas des viscosités non-bornées)**.** Les résultats de stabilité présentés sont également valables dans le cas d'une viscosité non-bornée.

La preuve de convergence pose plus de difficultés. Le passage à la limite dans le second membre de l'équation de bilan de k dans le cas des problèmes elliptiques à viscosités non-bornées, peut être effectué en utilisant le Théorème 3.6.1. Ce résultat de densité des fonctions $C_c^\infty(\Omega)$ dans le Sobolev à poids $W^{1,2}(\Omega)$), permet de prendre $\boldsymbol{u}$ comme fonction test dans l'équation pour montrer la convergence forte de $(\lambda^{(m)})^{1/2}\boldsymbol{\nabla}u^{(m)}$ dans $L^2(\Omega)$. Dans le cas du problème de Stokes, cette preuve n'est pas valable car le passage à la limite dans le terme de pression pose problème. Cette dernière n'appartient pas à $L^2(\Omega)$ dans le cas où λ est non-borné, mais seulement à $L^\alpha(\Omega)$, avec $\alpha\in[1,4d/3d-2)$.

Les éléments de preuves présentés sont à considérer comme un premier pas vers une preuve de convergence, dans la mesure où les mêmes méthodes s'appliquent si l'on est en mesure de montrer un résultat de densité similaire pour un espace de Sobolev à poids à divergence nulle (dans ce cas le terme de pression s'annule).

1.6 Vers le problème instationnaire : approximation volumes finis de l'équation de convection–diffusion avec second membre L^1

Dans la section précédente on a montré une résultat de convergence pour un problème modèle constitué des équations de Stokes stationnaires incompressibles et d'une équation de convection–diffusion stationnaire présentant un second membre de type production turbulente. Grâce à l'estimation d'énergie de l'équation de bilan de quantité de mouvement, on peut montrer que ce second membre appartient à $L^1(\Omega)$. Si la convergence du schéma volumes finis usuel a été abordé pour le cas du Laplacien par Gallouët–Herbin [13] et Herbin–Bradji [5], il semble que l'analyse du problème instationnaire n'ait pas été abordé dans le cadre discret. Le problème envisagé pour poursuivre l'étude théorique de l'algorithme utilisé dans cette thèse est donc l'analyse de convergence de l'équation de convection–diffusion volumes finis instationnaire avec un second membre appartenant à $L^1(\Omega)$:

$$\begin{aligned} &\partial_t u + \boldsymbol{\nabla}\cdot(u\,\boldsymbol{v}) - \Delta u = f && \text{in } \Omega\times(0,T),\\ &u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) && \text{a.e. in } \Omega,\\ &u(\boldsymbol{x},t) = 0 && \text{a.e. in } \partial\Omega\times(0,T), \end{aligned} \tag{1.28}$$

avec Ω un ouvert borné connexe de $\mathbb{R}^d$, $d=2,3$, $f\in L^1(\Omega\times(0,T))$ et $u_0\in L^1(\Omega)$.

Le champ de vitesse $\boldsymbol{v}$ est supposé solénoïdal, s'annule au bord du domaine Ω et suffisamment régulier :

$$\begin{aligned} &\boldsymbol{v}\in C^1(\bar\Omega\times[0,T]),\\ &\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t) = 0,\ \forall(\boldsymbol{x},t)\in\Omega\times(0,T),\\ &\boldsymbol{v}(\boldsymbol{x},t) = 0,\ \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T). \end{aligned} \tag{1.29}$$

Le schéma numérique proposé s'écrit :

$$\begin{aligned}&\forall K \in \mathcal{M},\ \text{for } 0 \le n < N,\\ &\frac{|K|}{\delta t}\left(u_K^{n+1} - u_K^n\right) + \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2}\, u_\sigma^{n+1} + \sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\left(u_K^{n+1} - u_L^{n+1}\right)\\ &\qquad + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma} u_K^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \int_K f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\end{aligned} \tag{1.30}$$

tel que pour $\sigma \in \mathcal{E}_{\mathrm{int}}$ et $0 \le n \le N$, on note $\mathrm{v}_{K,\sigma}^{n+1/2}$ l'approximation du champ de vitesse est défini par :

$$\mathrm{v}_{K,\sigma}^{n+1/2} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \int_{\sigma=K|L} \boldsymbol{v}(\boldsymbol{x},t)\cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}t \tag{1.31}$$

et la discrétisation de u sur les faces internes est la discrétisation décentrée amont.

L'analyse de convergence pour l'approximation volumes finis du Laplacien a été abordée par Gallouët–Herbin [13] et Bradji–Herbin [5]. De manière similaire au cas continu, on peut montrer que les solutions approchées sont contrôlées en norme discrète $\mathrm{W}^{1,q}(\Omega)$ pour $q < d/(d-1)$ dans le cas stationnaire. La compacité d'une famille de solutions $\{u^{(m)}\}_{m\in\mathbb{N}}$ est obtenue de manière usuelle par un théorème de Kolmogorov discret, moyennant l'estimation uniforme (par rapport à m) des translatées en espace dans $\mathrm{L}^q(\Omega)$. Néanmoins dans le cas instationnaire plusieurs difficultés apparaissent.

Si des résultats de compacité dans $\mathrm{L}^2(\Omega)$ en espace et en temps peuvent être obtenus par les mêmes techniques dans le cadre de l'analyse des équations de Navier–Stokes [12, 14], celles-ci ne sont pas transposables aux problèmes avec second membre irrégulier. En effet, la preuve de compacité par un théorème de Kolmogorov discret en temps requiert l'estimation des translatées en temps de la solution. La technique détaillée dans l'annexe A.a consiste à développer les translatées en temps en la somme de sauts de solution : cette décomposition fait apparaître des termes de bord (en temps) ainsi qu'une somme de termes de la forme $\|\partial_{t,\mathcal{M}} u\|_* \|u\|^*$, où $\|\cdot\|_*$ est la norme duale de $\|\cdot\|^*$. Dans le cas L^2, l'estimation des translatées en temps est directe car la dérivée temporelle de u appartient à l'espace dual de u (L^2 étant l'espace pivot) : typiquement u appartient à $\mathrm{L}^2(O,T;H_0^1(\Omega))$ et $\partial u/\partial t$ appartient $\mathrm{L}^2(O,T;\mathrm{H}^{-1}(\Omega))$. Mais dans le cas de l'équation à donnée L^1, on montre que la solution est contrôlée en norme discrète $\mathrm{L}^q(O,T;\mathrm{W}_0^{1,q}(\Omega))$, pour $1 \le q < (d+2)/(d+1)$ tandis que la dérivée en temps est contrôlée en norme discrète $\mathrm{L}^1(O,T;\mathrm{W}^{-1,q}(\Omega))$. Si l'argument de dualité ne peut pas être utilisé, l'estimation des translatées en temps peut être obtenue en montrant le lemme d'Aubin–Simon discret suivant :

Théorème 1.6.1. *Soit $(u^{(m)})_{m\in\mathbb{N}}$ une suite de fonctions discrètes, telle que $m \in \mathbb{N}$, $u^{(m)}$ est une fonction appartenant à l'espace $\mathrm{H}_{\mathcal{M}}^{(m)}$ associé au maillage $\mathcal{M}^{(m)}$ et au pas de temps $\delta t^{(m)}$. On suppose que la suite de maillages $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ est régulière au sens où il existe un nombre réel strictement positif ξ_0 tel que $\xi_{\mathcal{M}}^{(m)} \ge \xi_0$, $\forall m \in \mathbb{N}$ avec $\xi_{\mathcal{M}}$ défini par :*

$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \quad \xi_{\mathcal{M}} \le \frac{d_{K,\sigma}}{d_\sigma}, \quad \textit{and} \quad \xi_{\mathcal{M}} \le \frac{d_{K,\sigma}}{h_K}. \tag{1.32}$$

et que $h_{\mathcal{M}}^{(m)}$ et $\delta t^{(m)}$ tendent vers zéro quand m tend vers $+\infty$. On suppose qu'il existe trois réels $C > 0$, $q \ge 1$ et $r > 1$ tels que :

$$\forall m \in \mathbb{N}, \quad \sum_{n=1}^{N^{(m)}} \delta t^{(m)}\ \|(u^{(m)})^n\|_{1,q,\mathcal{M}} \le C, \qquad \sum_{n=2}^{N^{(m)}} \delta t^{(m)}\ \|(\partial_{t,\mathcal{D}}(u)^{(m)})^n\|_{-1,r,\mathcal{M}} \le C.$$

Alors la suite $(u^{(m)})_{m\in\mathbb{N}}$ converge à une sous-suite près dans $\mathrm{L}^1(\Omega\times(0,T))$ vers une fonction $u \in \mathrm{L}^q(0,T;\mathrm{W}_0^{1,q})$.

On peut ainsi montrer la compacité d'une famille de solution discrètes $\{u^{(m)}\}_{m\in\mathbb{N}}$ quand $u^{(m)}$ est contrôlé en norme discrète $\mathrm{L}^1(O,T;\mathrm{W}_0^{1,q}(\Omega))$ et sa dérivée en temps est estimée dans la norme $\mathrm{L}^1(O,T;\mathrm{W}^{-1,r}(\Omega))$, tel que r n'est pas forcément égal à q'.

On doit être toutefois prudent sur le sens à donner à la notion de compacité L^p en discret : une famille de fonctions discrètes $\{u^{(m)}\}_{m\in\mathbb{N}}$ appartenant chacune à $\mathrm{H}^{(m)}_{\mathcal{M}}$, est relativement compacte dans $\mathrm{L}^p(\Omega)$ au sens du théorème de Kolmogorov : c'est-à-dire que l'on peut extraire une sous-suite notée $\left(u^{(m)}\right)_{m\in\mathbb{N}}$ (qui est la suite des solutions approchées avec $h^{(m)}_{\mathcal{M}} \to 0$ quand $m \to 0$) telle que $u^{(m)} \to u$, et il existe $C > 0$, tel que $\|u\|_{1,p,\mathcal{M}} \leq C$ (ce qui est une conséquence de l'estimation des translatées).

Dans ce chapitre, on montre la convergence du schéma numérique au sens où, considérant une suite de solution discrète $\left(u^{(m)}\right)_{m\in\mathbb{N}}$ avec un maillage admissible $\mathcal{M}^{(m)}$ et un pas de temps $\delta t^{(m)}$, celle-ci converge dans $\mathrm{L}^1(\Omega \times (0,T))$ vers une solution du problème continu au sens précisé par le théorème suivant :

Théorème 1.6.2. *Soit $(u^{(m)})_{m\in\mathbb{N}}$ une suite de solutions discrètes, sur un maillage $\mathcal{M}^{(m)}$ et avec un pas de temps $\delta t^{(m)}$. On suppose que la suite de maillages $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ est régulière, au sens où la famille du paramètre de régularité $(\xi^{(m)}_{\mathcal{M}})_{m\in\mathbb{N}}$ satisfait $\xi^{(m)}_{\mathcal{M}} \geq \xi_0 > 0$, $\forall m \in \mathbb{N}$, et que $h^{(m)}_{\mathcal{M}}$ et $\delta t^{(m)}$ tendent vers zéro quand m tend vers $+\infty$.*

Alors, à une sous-suite près, la suite $(u^{(m)})_{m\in\mathbb{N}}$ converge dans $\mathrm{L}^1(\Omega \times (0,T))$ vers une fonction $u \in \mathrm{L}^q(0,T;\mathrm{W}^{1,q}_0(\Omega))$, pour tout $q \in [1,(d+2)/(d+1))$, qui est une solution faible du problème continu, au sens où :

$$u \in \cup_{1\leq q<(d+2)/(d+1)}\ \mathrm{L}^q(0,T;\mathrm{W}^{1,q}_0(\Omega))$$

et, $\forall \varphi \in \mathrm{C}^\infty_c(\Omega \times [0,T))$:

$$\begin{aligned}
&-\int_{\Omega\times(0,T)} u(\boldsymbol{x},t)\,\partial_t\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\Omega\times(0,T)} u\,\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_{\Omega\times(0,T)} \nabla u(\boldsymbol{x},t)\cdot\nabla\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{\Omega\times(0,T)} f\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega u_0(\boldsymbol{x})\,\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Estimations a priori : on montre que u est contrôlé en norme discrète $\mathrm{L}^\infty(O,T;\mathrm{L}^1(\Omega))$ et $\mathrm{L}^q(O,T;\mathrm{W}^{1,q}_0(\Omega))$, pour $q \in [1,(d+2)/(d+1))$ uniformément par rapport au paramètre de régularité de maillage.

La première estimation tire parti du résultat de stabilité de l'opérateur de convection étendu proposé au théorème 1.6.4 dont la preuve détaillée se situe en annexe du chapitre 4. Supposons que le bilan de masse discret est satisfait, au sens où il existe deux familles de réels $(\rho_K)_{K\in\mathcal{M}}$ et $(\rho^*_K)_{K\in\mathcal{M}}$ vérifiant :

$$\forall K \in \mathcal{M},\quad \rho_K > 0,\quad \rho^*_K > 0 \quad \text{et} \qquad |K|\,\frac{\rho_K - \rho^*_K}{\delta t} + \sum_{\sigma=K|L} F_{\sigma,K} = 0 \tag{1.33}$$

où $(F_{\sigma,K})_{K\in\mathcal{M},\ \sigma=K|L}$ est une quantité conservative associée à la face σ et au volume de contrôle K, *i.e.* telle que $F_{\sigma,K} = -F_{\sigma,L}$, $\forall\sigma = K|L$, et qui constitue une approximation du flux massique à la face σ.

Alors le résultat suivant, démontré dans l'article [16], est vérifié :

Théorème 1.6.3 (Stabilité de l'opérateur de convection volumes finis). *Soient $(\rho^*_K)_{K\in\mathcal{M}}$, $(\rho_K)_{K\in\mathcal{M}}$ et $(F_{\sigma,K})_{K\in\mathcal{M},\ \sigma=K|L}$ trois familles de réels qui vérifient la condition* (1.33). *Soient $(s^*_K)_{K\in\mathcal{M}}$ et $(s_K)_{K\in\mathcal{M}}$ deux familles de réels. Pour toute face interne $\sigma = K|L$, on définit s_σ soit par $s_\sigma = \frac{1}{2}(s_K + s_L)$, soit par $s_\sigma = z_K$ si $F_{\sigma,K} \geq 0$ et $s_\sigma = z_L$ autrement. Le premier choix est nommé "centré", le second "décentré amont" ou "upwind". Dans les deux cas, on a l'inégalité de stabilité suivante :*

$$\sum_{K\in\mathcal{M}} s_K \left[\frac{|K|}{\delta t}\,(\rho_K\,s_K - \rho^*_K\,s^*_K) + \sum_{\sigma=K|L} F_{\sigma,K}\,s_\sigma\right] \geq \frac{1}{2}\,\sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\,\left[\rho_K\,s_K^2 - \rho^*_K\,{s^*_K}^2\right]$$

On montre que ce résultat peut être étendu au cas où l'on teste l'opérateur de convection volumes finis contre $\phi(s)$, où ϕ est une fonction à valeurs réelles concave. Le théorème précédent correspond au cas $\phi(s) = s$.

Théorème 1.6.4. *Soient $(\rho^*_K)_{K\in\mathcal{M}}$, $(\rho_K)_{K\in\mathcal{M}}$ et $(F_{\sigma,K})_{K\in\mathcal{M},\ \sigma=K|L}$ trois familles de réels qui vérifient la condition* (1.33). *Soient $(s^*_K)_{K\in\mathcal{M}}$ et $(s_K)_{K\in\mathcal{M}}$ deux familles de réels. Soit Φ une fonction convexe positive* *monotone sur $\mathbb{R}^+$ et ϕ sa dérivée. On suppose que pour toute face interne $\sigma = K|L$, s_σ est l'approximation décentrée amont. Alors l'inégalité suivante est vérifiée :*

$$\sum_{K\in\mathcal{M}} \phi(s_K) \left[\frac{|K|}{\delta t} \left(\rho_K\, s_K - \rho_K^*\, s_K^*\right) + \sum_{\sigma=K|L} F_{\sigma,K}\, s_\sigma \right] \geq \sum_{K\in\mathcal{M}} |K| \rho_K \frac{\Phi(s_K) - \Phi(s_K^*)}{\delta t}$$

L'équation discrète est testée contre la fonction $\phi'(u)$ définie par :

$$\forall y \in \mathbb{R}, \qquad \phi'(y) = \int_0^y \frac{1}{1+|s|^\theta}\, \mathrm{d}s, \qquad \text{and} \quad \phi(y) = \int_0^y \phi'(s)\, \mathrm{d}s \tag{1.34}$$

La fonction ϕ est positive et convexe, tandis que sa dérivée est négative sur $\mathbb{R}^-$ et positive sur $\mathbb{R}^+$ et bornée. Par argument de convexité, le terme de diffusion est positif tandis que le terme de convection est minoré grâce au lemme de stabilité, ce qui permet d'obtenir le contrôle de u par les données.

La seconde estimation s'inspire de la preuve dans le cas continu de Boccardo–Gallouët [2] et utilise l'estimation dans le cas elliptique dans [5], qui montre que u est borné en norme discrète $W^{1,p}(\Omega)$, pour $1 \leq p < 2$. Elle repose essentiellement sur le fait que l'on peut faire apparaître la norme discrète $L^q(0,T,L^{q^*}(\Omega))$, $q^* = dq/(d-q)$ par une inégalité de Hölder généralisée et utiliser une inégalité de Sobolev discrète pour contrôler celle-ci. L'exposant q est donc déterminé par un argument d'exposant critique de l'injection de Sobolev. De même que dans le cas continu, on montre que $q < (d+2)/(d+1)$.

Enfin, on montre en utilisant l'équation et l'estimation précédente que la dérivée discrète est contrôlée en norme $L^q(O,T;W^{1,-q}(\Omega))$

Compacité : Comme on l'a introduit, les estimations *a priori* étant plus faibles que L^2, la compacité de la famille de solutions approchées est obtenue dans $L^1(\Omega\times[0,T])$ en montrant un équivalent discret du lemme d'Aubin–Simon 1.6.1. Celui-ci peut être vu comme une extension du théorème de Kolmogorov, et est montré de manière similaire grâce aux estimations uniformes des translatées en espace et en temps dans $L^1(\Omega\times[0,T])$.

La difficulté principale de la preuve de celui-ci réside dans l'estimation des translatées en temps. On montre que l'on peut estimer les translatées en temps en norme $L^1(\Omega\times[0,T])$ si une inégalité de Lions est vérifiée en discret. De manière similaire au cas continu, on montre donc dans un premier temps un équivalent discret du Lemme de Lions [4] pour les normes $L^q(\Omega)$, $W^{1,q}(\Omega)$ et $W^{1,-r}(\Omega)$ discrètes, avec $1 \leq q, r < \infty$. Alors, pour tout nombre réel $\delta > 0$, il existe une constante $C(\delta)$ telle que l'inégalité suivante est vérifiée :

$$\|v\|_{L^q(\Omega)} \leq \delta\, \|v\|_{1,q,\mathcal{M}} + C(\delta)\, \|v\|_{-1,r,\mathcal{M}}$$

En utilisant cette inégalité avec $v = u(\boldsymbol{x}, t+\tau) - u(\boldsymbol{x}, t)$ puis en intégrant temps, on peut alors borner chacun des termes au second membre de cette inégalité en utilisant respectivement les estimations *a priori* de u et sa dérivée en temps.

Passage à la limite : Le passage à la limite dans les termes de diffusion et de convection utilise les techniques usuelles de l'analyse des schémas volumes finis, pour lesquelles on peut se référer à [26].

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Chapitre 2

A monotone scheme for two-equation turbulence models

A MONOTONE SCHEME FOR TWO-EQUATION TURBULENCE MODELS

Abstract.In this paper which is an extended version of an article submitted to the *Finite Volumes for Complex Applications V* conference, we design a finite-volume based numerical scheme for the solution of the nonlinear balance equations of RNG variants of the well-known $k-\varepsilon$ model and the $\overline{v^2}-f$ system encountered in the $k-\varepsilon-\overline{v^2}-f$ model which can be seen as an extension of first order turbulence models. In this class of models, the description of the turbulence relies on two variables, the turbulent kinetic energy k and its dissipation rate ε, which, for physical reasons, must remain positive. When standard upwinding techniques for the convection terms are used, the presented scheme is proved to preserve the positivity of these two unknowns, and, through a topological degree argument, to admit at least a solution. Moreover, the computation of the values of k and ε in the near-wall regions requires that the mesh is highly refined if no treatment if enforced, since the characteristic scales of the solution decay towards the order of the length of molecular dissipation as the distance to the wall tends to zero. We described here for the sake of completness of the dissertation, the underlying ideas to design of wall-laws in the case of the $k-\varepsilon$ and $k-\varepsilon-\overline{v^2}-f$ models, which amounts to enforcing Dirichlet boundary conditions for the turbulent scales at walls, which are solution to an asymptotic model. The study led to the reevaluation of some model constants, comparing to the literature, in the case of the $k-\varepsilon-\overline{v^2}-f$ model. Finally, a numerical convergence study has been performed to assess the properties of the scheme: a first order convergence rate in both time and space is verified in the case of upwind fluxes and when using a MUSCL discretization for the approximation of the convection terms, the scheme becomes of second order in space.

2.1 Introduction

The problem addressed in this paper is the numerical treatement of turbulence equations of the widely-used $k-\varepsilon$ model, in a fractional step scheme for the solution of low Mach number forced or natural convection flows, as addressed by the fire simulation code ISIS developed at IRSN. The flow is supposed to obey the Favre-averaged Navier-Stokes equations, posed on an open bounded subset of $\mathbb{R}^d$, $d=2,3$, and over a finite time interval $(0,T)$:

$$\left|\begin{array}{l} \partial_t(\rho\boldsymbol{u})+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})= \\ \qquad\qquad \boldsymbol{\nabla}\cdot\big((\mu+\mu_t)(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}^t\boldsymbol{u}-\dfrac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathbf{I})\big)-\boldsymbol{\nabla}p+\rho\,\boldsymbol{g} \\ \partial_t(\rho)+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})=0 \end{array}\right. \tag{2.1}$$

where ρ stands for the fluid density, whose value is positive and bounded, u is the mean velocity field, μ and μ_t are two positive scalar fields called respectively the laminar and turbulent viscosity and $\boldsymbol{g}$ stands for a forcing term. The density is supposed to vary with space and time, as a function of the fluid temperature and composition, which are governed by additional balance equations which are not defined here.

The $k-\varepsilon$ model relies on two positive scalar fields k and ε, respectively the kinetic turbulent energy and its dissipation rate, to describe the turbulent characteristics of the flow. These variables are the solution to a system of two coupled nonlinear parabolic balance equations, which reads:

$$\left|\begin{array}{l} \partial_t(\rho k)+\boldsymbol{\nabla}\cdot(\rho k\boldsymbol{u})-\boldsymbol{\nabla}\cdot\Big(\big(\mu+\dfrac{\mu_t}{\sigma_k}\big)\boldsymbol{\nabla}k\Big)=\mathbf{P}+\mathbf{G}-\rho\varepsilon \\ \partial_t(\rho\varepsilon)+\boldsymbol{\nabla}\cdot(\rho\varepsilon\boldsymbol{u})-\boldsymbol{\nabla}\cdot\Big(\big(\mu+\dfrac{\mu_t}{\sigma_\varepsilon}\big)\boldsymbol{\nabla}\varepsilon\Big) \\ \qquad\qquad =\dfrac{\varepsilon}{k}\Big(C_{\varepsilon 1}\mathbf{P}+C_g\mathbf{G}-\rho C_{\varepsilon 2}\varepsilon\Big)+\mathbf{S}_{\rm rng} \end{array}\right. \tag{2.2}$$

This turbulence model involves a set of positive constants: the so-called turbulent Prandtl-Schmidt numbers σ_k and σ_ε, and the empirical constants C_μ, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$. The source terms include the turbulent production $\mathbf{P}$, a buoyancy term $\mathbf{G}$ which may become negative, and a correction to the standard $k-\varepsilon$ model denoted by $\mathbf{S}_{\rm rng}$. The turbulent production $\mathbf{P}$ is a function of the mean shear stress in the flow and reads in the compressible case:

$$\mathbf{P} = \max\left[\frac{1}{2}\mu_t\ \|\boldsymbol{\nabla u} + \boldsymbol{\nabla}^t\boldsymbol{u}\|^2 - \frac{2}{3}\Big(\rho k + \mu_t \boldsymbol{\nabla}\cdot\boldsymbol{u}\Big)\boldsymbol{\nabla}\cdot\boldsymbol{u},\ \ 0\right]$$

The coupling between the k-equation and the ε-equation arises, on the one hand, in the diffusion and production terms, through the definition of the turbulent viscosity as a non-linear function of k and ε, and, on the other hand, from the sink term in the ε-equation. Under the so-called "Prandtl-Kolmogorov hypothesis", the turbulent viscosity reads:

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon} \tag{2.3}$$

where C_μ is a positive constant.

The RNG-variants of the $k-\varepsilon$ model differ from the standard model by a new set of constants and the additional source term $\mathbf{S}_{\rm rng}$ in the ε-equation. Both are obtained from the application of renormalization group techniques to the Navier-Stokes equations; the additional source term models a correction of the ε–destruction term, making it dependent on a positive parameter η, which represents the ratio of the turbulent relaxation time over a time-scale for the mean flow. We give here, as an example, its expression for the two most common variants. For the first one [11], this additional source term reads:

$$\eta = \left[\frac{\mathbf{P}}{\rho C_\mu \varepsilon}\right]^{1/2}, \qquad \mathbf{S}_{\rm rng} = -\rho C_\eta(\eta)\mathbf{P}\frac{\varepsilon}{k}, \qquad C_\eta(\eta) = \frac{\eta(1-\eta/\eta_0)}{1+\beta\eta^3}$$

and, for the second one [7]:

$$\eta = \frac{1}{\sqrt{2}}\ \|\boldsymbol{\nabla u} + \boldsymbol{\nabla}^t\boldsymbol{u}\|\ \frac{k}{\varepsilon} \qquad \mathbf{S}_{\rm rng} = -\rho C_\mu C_\eta(\eta)\eta^2\frac{\varepsilon^2}{k} + \rho C_{\varepsilon 3}(\eta)\varepsilon\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

The function of η, $C_\eta(\eta)$, may be either positive or negative and the function $C_{\varepsilon 3}(\eta)$ is such that $C_{\varepsilon 3}(\eta)\boldsymbol{\nabla}\cdot\boldsymbol{u} \geq 0$. The constants involved in these expressions are gathered in Table 2.1.

In this paper, we build a finite volume scheme for the solution of system (2.2), which ensures the positivity of both k and ε (section 2.2). This yields a non-linear discrete problem which is proven to admit at least one solution, by a topological degree argument (section 2.3). Finally, we investigate the convergence properties of this scheme by numerical experiments (section 2.7). This scheme retains as unknown the primitive variables k and ε, although different choices (typically, products of the form $k^{\xi_1}\varepsilon^{\xi_2}$, with ξ_1 and ξ_2 two given real numbers) have been investigated in the literature [5]. The present study applies, possibly with minor modifications, to the standard $k-\varepsilon$ model as well as the two RNG variants above described, all these models being implemented in the ISIS code. However, for the sake of conciseness, the exposition is restricted here to the first RNG variant.

2.2 A numerical scheme for the $k-\varepsilon$ RNG model

The equations of system (2.2) are both solved by using a finite volume discretization on an admissible mesh $\mathcal{M}$ (in the sense of [2], Chapter 3) of the computational domain Ω. This mesh is composed of a family $\mathcal{M}$ of control volumes, which are convex disjoint polygons ($d=2$) or polyhedrons ($d=3$) included in Ω and such that $\bar\Omega = \bigcup_{K\in\mathcal{M}} \bar K$. For each neighbouring control volume L of $K \in \mathcal{M}$, $\sigma = K|L$ denotes the

	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	C_μ	σ_k	σ_ε	β	η_0
first variant	1.42	1.68	0.0845	0.719	0.719	0.015	4.38
second variant	1.42	1.68	0.0837	0.719	0.719	0.012	4.38

Table 2.1: Value of the constants for the considered RNG variants of the $k-\varepsilon$ model.

common edge of K and L. Finally, for the discretization of diffusion terms, we suppose that we are able to build a family $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ of points of Ω such that $x_K \in \bar{K}$ for all $K \in \mathcal{M}$ and, if $\sigma = K|L$, $x_K \neq x_L$ and the straight line going through x_K and x_L is orthogonal to σ [2, Chapter 3, Figure 3.2]. By $|K|$ and $|\sigma|$, we denote hereafter the measure of the control volume K and of the edge or face σ, respectively. For any control volume K and edge or face σ of K, we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. For any edge or face σ, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, if σ separates the two control volumes K and L (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if σ is included in the boundary. For each control volume K, $\mathcal{E}(K)$ denotes the set of edges or faces of K, and $\mathcal{E}_{\text{ext}}$ is the set of edges or faces lying on the boundary.

To prove the positivity of the fields k and ε, we use hereafter an algebraic argument which we recall here: if A is a strictly diagonally dominant matrix whose diagonal entries are positive and off-diagonal entries are non-positive, then A is a non-singular M-matrix and thus the inverse matrix A^{-1} is non-negative [10].

The discrete finite volume operators which are needed to describe the proposed scheme are defined by, $\forall K \in \mathcal{M}$:

$$\begin{aligned}
[\boldsymbol{\nabla}_{\mathcal{D}}\cdot a]_K &= \frac{1}{|K|}\sum_{\sigma=K|L} |\sigma|\ a_\sigma\cdot n_\sigma \\
-[\boldsymbol{\Delta}_{\mathcal{D},\lambda}(s)]_K &= \frac{1}{|K|}\Bigg[\sum_{\sigma=K|L}\lambda_\sigma\frac{|\sigma|}{d_\sigma}(s_K-s_L)+\sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\text{ext}}}\lambda_\sigma\frac{|\sigma|}{d_\sigma}s_K\Bigg]
\end{aligned}$$

where a and s are respectively a generic vector valued and scalar unknown field, $\boldsymbol{\nabla}_{\mathcal{D}}\cdot\ a$ stands for the approximation of $\boldsymbol{\nabla}\cdot a$, supposing that the a vanishes on the boundary, and $\boldsymbol{\Delta}_{\mathcal{D},\lambda}(s)$ stands for the approximation of $\boldsymbol{\nabla}\cdot\big(\lambda\boldsymbol{\nabla}s\big)$ with homogeneous Dirichlet boundary conditions, where λ_σ stands for the value at the centre of the edge, approximated, in the pratice, either by the arithmetic or harmonic mean of λ_K and λ_L. Let $(t^n)_{\,0\le n\le N}$ be a uniform partition of the time-interval $(0,T)$ and δt be the constant time step $\delta t = t^{n+1} - t^n,\ 0 \le n < N$.

With these notations, the proposed Euler-implicit time-discretization of the $k-\varepsilon$ RNG system reads:

$$\left|\begin{aligned}
&\frac{\rho^n k^{n+1}-\rho^{n-1}k^n}{\delta t}+\boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n k^{n+1}\boldsymbol{u}^n)-\boldsymbol{\Delta}_{\mathcal{D},\mu+(\mu_t/\sigma_k)}(k^{n+1})\\
&\qquad=\mathbf{P}^n+[\mathbf{G}^n]^+-([\mathbf{G}^n]^-+\rho^n|\varepsilon^{n+1}|)\frac{k^{n+1}}{\max(k^*,k^n)}\\
&\frac{\rho^n\varepsilon^{n+1}-\rho^{n-1}\varepsilon^n}{\delta t}+\boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n\varepsilon^{n+1}\boldsymbol{u}^n)-\boldsymbol{\Delta}_{\mathcal{D},\mu+(\mu_t/\sigma_\varepsilon)}(\varepsilon^{n+1})\\
&\qquad=\Big(C_{\varepsilon1}\mathbf{P}^n+C_g[\mathbf{G}^n]^+\Big)\ \frac{\varepsilon^n}{\max(k^*,k^n)}-\rho^n C_{\varepsilon2}\ \frac{|\varepsilon^{n+1}|\varepsilon^{n+1}}{\max(k^*,k^n)}\\
&\qquad-\rho^n\mathbf{P}^n\left[C_\eta(\eta^n)^+\ \frac{\varepsilon^{n+1}}{\max(k^*,k^n)}-C_\eta(\eta^n)^-\frac{\varepsilon^n}{\max(k^*,k^n)}\right]
\end{aligned}\right. \tag{2.4}$$

where, for any real number s, $s^+ = \max(s, 0)$ and $s^- = -\min(s, 0)$ so that $s^+ \ge 0$, $s^- \ge 0$ and $s = s^+ - s^-$ and k^* stands for a residual value ($k^* = 10^{-10}$ here, this value being never reached in the presented applications). The quantities $(\rho^n\zeta^{n+1}\boldsymbol{u}^n)_\sigma$, with $\zeta = k$ or $\zeta = \varepsilon$, appearing in the convection terms are evaluated as $(\rho^n\zeta^{n+1}\boldsymbol{u}^n)_\sigma = (\rho^n\boldsymbol{u}^n)_\sigma\ (\zeta^{n+1})_\sigma$, where $(\zeta^{n+1})_\sigma$ stands for the upwind (with respect to $(\rho^n\boldsymbol{u}^n)_\sigma$) approximation of ζ^{n+1} on σ, and the mass fluxes $(\rho^n\boldsymbol{u}^n)_\sigma$ are supposed to satisfy the following relation:

$$\frac{\rho^n-\rho^{n-1}}{\delta t}+\boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n\boldsymbol{u}^n)=0 \tag{2.5}$$

which means that $(\rho^n\boldsymbol{u}^n)_\sigma$ must be evaluated in the same way as in the mass balance at the previous time step, whatever its discretization in this latter equation may be. The time semi-discretization of the source terms is designed to ensure the positivity of k and ε, following guidelines described in [6]: positive source terms (as $\mathbf{P}^n$ or $[\mathbf{G}^n]^+$) are left unchanged and negative source terms are absorbed in the diagonal of the discrete operator by multiplying them by ζ^{n+1}/ζ^n. Terms as $(\zeta^{n+1})^2$ are recast as $|\zeta^{n+1}|\,\zeta^{n+1}$, without consequences for the consistency of the scheme since we prove in the following that $\zeta^{n+1} > 0$. Finally the diffusion coefficient (which, since μ_t depends on k and ε, varies with the time) is evaluated at the previous time step without any significant influence on the stability of the scheme, at least in the numerical tests performed up to now.

Proposition 2.2.1. *Let us suppose that the initial condition* (k^0, ε^0) *is positive. Then, for all* $0 \leq n < N$ *and without restriction on the time-step* δt*, any possible solution* $(k^{n+1}, \varepsilon^{n+1})$ *of discrete system* (2.4) *is positive.*

Proof. The technique used in this proof is to recast the scheme (2.4) under the form:

$$\begin{pmatrix} A_{k,k} & 0 \\ 0 & A_{\varepsilon,\varepsilon} \end{pmatrix} \begin{pmatrix} k^{n+1} \\ \varepsilon^{n+1} \end{pmatrix} = \begin{pmatrix} S_k \\ S_\varepsilon \end{pmatrix}$$

where each block A_{kk} or $A_{\varepsilon\varepsilon}$ of the matrix depends on the unknowns $(k^{n+1}, \varepsilon^{n+1})$, but is an M-matrix whatever their value may be (because it satisfies the assumptions given in section 2.2), and both blocks S_k and S_ε of the right hand side are positive. One proceeds by induction, supposing that k^n and ε^n are positive. Since the proof for both blocks of the system are quite similar, we only deal here with the first one.

Thanks to the definition of the discrete operators, the k-equation at time t^{n+1} for a given control volume $K \in \mathcal{M}$ can be written as follows:

$$\begin{aligned} &\left[\frac{|K|\ \rho_K^n}{\delta t} + \sum_{\sigma=K|L} |\sigma| \left((\rho^n \boldsymbol{u}^n)_\sigma^+ + \frac{\alpha_\sigma^n}{d_\sigma} \right) + |K|\ \frac{[\mathbf{G}_K^n]^- + \rho_K^n |\varepsilon_K^{n+1}|}{\max(k^*, k_K^n)} \right] k_K^{n+1} \\ &\quad - \sum_{\sigma=K|L} \left[|\sigma| (\rho^n \boldsymbol{u}^n)_\sigma^- + \frac{\alpha_\sigma^n}{d_\sigma} \right] k_L^{n+1} = |K| \left[\mathbf{P}_K^n + [\mathbf{G}_K^n]^+ + \frac{\rho_K^{n-1} k_K^n}{\delta t} \right] \end{aligned} \tag{2.6}$$

where α_σ^n stands for the diffusion. If this equation is considered as a linear system for k^{n+1}, in the corresponding matrix $A_{k,k}$, the first term yields a positive diagonal entry, while, in the second one, each term of the summation over the neighbouring control volumes yields a negative off-diagonal entry. Moreover the summation of all matrix elements in a row reads:

$$\frac{|K|\ \rho_K^n}{\delta t} + \sum_{\sigma=K|L} |\sigma| (\rho^n \boldsymbol{u}^n)_\sigma + |K|\ \frac{[\mathbf{G}^n]^- + \rho_K^n |\varepsilon_K^{n+1}|}{\max(k^*, k_K^n)}$$

The last fraction is non-negative, while the positivity of the sum of the first two terms holds provided that the mass balance is fulfilled; actually if the discrete divergence operator $\boldsymbol{\nabla}_{\mathcal{D}}\cdot$ is built upon the same fluxes approximation as the discrete counterpart of the second relation of (2.1), these terms read:

$$|K| \left(\frac{\rho_K^n}{\delta t} + [\boldsymbol{\nabla}_{\mathcal{D}} \cdot \rho^n \boldsymbol{u}^n]_K \right) = \frac{|K|\ \rho_K^{n-1}}{\delta t} \quad > 0$$

This is essentially the argument developed in [4]. The matrix $A_{k,k}$ is thus an M-matrix. As the right hand side of equation (2.6) is positive, $k^{n+1} > 0$. ☐

2.3 Existence of a solution

Lemma 2.3.1. *Let us suppose that* v^{n+1} *satisfies the following non-homogeneous reaction-convection-diffusion equation:*

$$\frac{\rho^n v^{n+1} - \rho^{n-1} v^n}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}} \cdot (\rho^n \boldsymbol{u}^n v^{n+1}) - \boldsymbol{\Delta}_{\mathcal{D}, \mu_v^n}(v^{n+1}) = g^+ - \alpha\, v^{n+1} \tag{2.7}$$

where $\mu_v^n > 0$ *and* $\alpha > 0$*, and let* $\bar{v} > \max(v^n) + \dfrac{\delta t}{\min(\rho^{n-1})} \max(g^+)$.

Then, assuming that the mass balance equation (2.5) *is verified,* $v^{n+1} < \bar{v}$.

Proof. Let $\bar{v}$ be a positive real number. When applied to a constant function, the diffusion operator vanishes and so does the advection term if mass balance (2.5) is verified. Thus:

$$\begin{aligned} &\frac{\rho^n (v^{n+1} - \bar{v})}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}} \cdot (\rho^n \boldsymbol{u}^n (v^{n+1} - \bar{v})) - \boldsymbol{\Delta}_{\mathcal{D}, \mu_v^n}(v^{n+1} - \bar{v}) \\ &\qquad + \alpha\, (v^{n+1} - \bar{v}) = \frac{\rho^{n-1}(v^n - \bar{v})}{\delta t} + g^+ - \alpha \bar{v} \end{aligned}$$

We can prove, as for proposition (2.2.1), that the matrix associated with this linear system is an M-matrix. Thus:

$$\frac{\rho^{n-1}(v^n - \bar{v})}{\delta t} + g^+ - \alpha\bar{v} < 0 \quad \Rightarrow \quad (v^{n+1} - \bar{v}) < 0$$

Let $\bar{v} > \max(v^n) + \dfrac{\delta t}{\min(\rho^{n-1})} \max(g^+)$; the first inequality above is verified, consequently, $v^{n+1} < \bar{v}$. □

Theorem 2.3.2 (Application of the topological degree (discrete case))**.** *Let be V a finite dimensional vector space defined over R, f a continuous application from V in V and a topological degree application $d : C^0(\bar{\Omega}, R^d) \times R^d \times R^d \to R$. Let suppose that there exists a function $F : [0,1] \times V \to V$ such that :*

1. $F(1, \cdot) = f(\cdot)$
2. *$\forall \xi \in [0,1]$, if v is solution of $F(v, \xi) = b$ then $v \in W$ such that $W = \{v \in W : \|v\| < R\}$ where R is a positive constant independent of ξ and $\|\cdot\|$ is a norm defined over W.*
3. $d_0 = d(F(0,\cdot), W, b) \neq 0$

then there exists a solution $v \in W$ to the equation $f(v) = b$.

Theorem 2.3.3. *Provided that the initial values for k and ε are positive, there exists a solution to the scheme* (2.4)*. In addition, any possible solution is positive.*

Proof. This result is proved by a topological degree argument. To this purpose, we identify the discrete space to $\mathbb{R}^N$, where N is the number of control volumes, and we define the function F, from $\mathbb{R}^N \times \mathbb{R}^N \times [0,1]$ to $\mathbb{R}^N \times \mathbb{R}^N$ by $F = (F_k, F_\epsilon)$ with:

$$F_k(k,\varepsilon,\xi) = \frac{\rho^n k^{n+1} - \rho^{n-1}k^n}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n k^{n+1}\boldsymbol{u}^n) - \boldsymbol{\Delta}_{\mathcal{D},\mu_k^n}(k^{n+1})$$
$$-\mathbf{P}^n - [\mathbf{G}^n]^+ + ([\mathbf{G}^n]^- + \xi\rho^n|\varepsilon^{n+1}|)\frac{k^{n+1}}{\max(k^*, k^n)}$$

This definition is obtained by multiplying the non-linear terms by the parameter $\xi \in [0,1]$ in the first equation of system (2.4). The function F_ϵ is obtained from the second equation of the same system in a similar way. The function F is continuous over $\mathbb{R}^N \times \mathbb{R}^N \times [0,1]$ and equation $F(k,\varepsilon,1) = 0$ is exactly the system (2.4).

On the one hand, lemma 2.3.1 yields a bound for any solution of $F(\xi,k,\varepsilon) = 0$, let us say $k < \bar{k}$ and $\varepsilon < \bar{\varepsilon}$ where both $\bar{k}$ and $\bar{\varepsilon}$ are independent of ξ. Moreover, the system $F(0,k,\varepsilon) = 0$ is linear and has a unique solution (its matrix is an M-matrix). Then, the topological degree of $F(\xi,k,\varepsilon)$ with respect to $\mathcal{O} = (0,\bar{k})^N \times (0,\bar{\varepsilon})^N$ and 0 is non-zero. Using Theorem 2.3.2 with $W = \mathcal{O}$ (which is correct since $\bar{k}$ and $\bar{\varepsilon}$ do not depend on ξ) and $b = 0$, we have $d(F(1,k,\varepsilon), W, 0) \neq 0$ since the degree is invariant by homotopy, which proves that the system of equation (2.4) admits a solution in $\mathcal{O}$. On the other hand, since the proof of Proposition 2.2.1 (performed in the case $\xi = 1$) holds for any $\xi \in [0,1]$, we find that any solution is positive. □

2.4 A uniqueness result

Theorem 2.4.1. *The solution to discrete Problem* (2.4) *is unique.*

Proof. Let (k_1,ε_1) and (k_2,ε_2) be two solutions to Problem (2.4) at the considered time step and (k^0,ε^0) the solution at the previous time step. According to (2.2.1) any of these solutions is positive. We denote by δk and $\delta\varepsilon$ respectively the quantities $k_1 - k_2$ and $\varepsilon_1 - \varepsilon_2$ and by $(\rho_K)_{K\in\mathcal{M}}$ and $(\rho_K^0)_{K\in\mathcal{M}}$, $(\boldsymbol{u}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ three families of real numbers.

The pair (k_1,ε_1) satisfies the following equations:

$$\frac{\rho k_1 - \rho^0 k^0}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho\boldsymbol{u}k_1) - \boldsymbol{\Delta}_{\mathcal{D},\mu_k}(k_1) = P^+ - \rho k_1\frac{|\varepsilon_1|}{k^0}$$
$$\frac{\rho\varepsilon_1 - \rho^0\varepsilon_1^0}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho\boldsymbol{u}\varepsilon_1) - \boldsymbol{\Delta}_{\mathcal{D},\mu_\varepsilon}(\varepsilon_1) = C_1 - C_2\varepsilon_1^2 - C_3\varepsilon_1$$

where the source terms are written under a simplified expression, with C_1, C_2 and C_3 three positive constants depending only on ρ, k^0 and ε^0. Substracting term by term the equations written for (k_1, ε_1) and (k_2, ε_2), we get for the second equation:

$$\frac{\rho\delta\varepsilon}{\delta t} + \nabla_{\mathcal{D}}\cdot(\rho\boldsymbol{u}\delta\varepsilon) - \boldsymbol{\Delta}_{\mathcal{D},\mu_\varepsilon}(\delta\varepsilon) = -C_2(\varepsilon_1^2 - \varepsilon_2^2) - C_3\delta\varepsilon$$

then, this latter can be put under the form of the following convection–diffusion–reaction equation with a positive reaction coefficient (since ε_1 and ε_2 are positive by Proposition 2.2.1):

$$\frac{\rho\delta\varepsilon}{\delta t} + \nabla_{\mathcal{D}}\cdot(\rho\boldsymbol{u}\delta\varepsilon) - \boldsymbol{\Delta}_{\mathcal{D},\mu_\varepsilon}(\delta\varepsilon) + (C_2(\varepsilon_1 + \varepsilon_2) + C_3)\delta\varepsilon = 0$$

which yields $\varepsilon_1 = \varepsilon_2$.

Returning to the first equation we have:

$$\frac{\rho\delta k}{\delta t} + \nabla_{\mathcal{D}}\cdot(\rho\boldsymbol{u}\delta k) - \boldsymbol{\Delta}_{\mathcal{D},\mu_k}(\delta k) = -\frac{\rho}{k^0}(k_1|\varepsilon_1| - k_2|\varepsilon_2|)$$

Since $\varepsilon_1 = \varepsilon_2$, the right-hand side is in fact linear with respect to δk and the resulting convection–diffusion–reaction equation (with $\rho|\varepsilon|/k^0 > 0$ as reaction coefficient) satisfies a maximum principle. Thus, $k_1 = k_2$, which proves the uniqueness of the solution. □

2.5 The $k - \varepsilon - \overline{v^2} - f$ model

An extension of the two-equation $k - \varepsilon$ turbulence model, which has been devised by Durbin [1] using arguments from the Reynolds Stress Model theory, has been studied to overcome the difficulties introduced by the stagnation point anomaly. The $\overline{v^2} - f$ model tries to enhance the near-wall prediction of turbulent flows by introducing an additional length scale $\overline{v^2}$ which behaves in the near-wall region as the fraction of turbulent kinetic energy due to the velocity fluctuations normal to the streamlines, as the stresses in the near-wall region reduces to a plane deformation. The use of this characteristic length enables to enforce a proper decay of turbulent viscosity in the equation, since the "normal component" of the turbulent kinetic energy is thus controlled ("kinematic blocking effect") and avoid the use of damping functions or high-Reynolds wall-functions. Moreover a function f is introduced to take into account of redistribution of the energy in the near-wall region by the pressure fluctuations ("pressure echo effect"), which is the solution of the so-called elliptique relaxation equation.

Accordingly the turbulent viscosity μ_t is computed by an algebraic relation similar to the Prandtl–Kolmogorov hypothesis (2.3) :

$$\mu_t = \rho C_\mu \overline{v^2} T \tag{2.8}$$

where $T = \min(k/\varepsilon, \tau_K)$, with τ_K the Kolmogorov time scale.

The length scale $\overline{v^2}$ is solution to a convection–diffusion balance equation similar to the balance equation of k:

$$\partial_t(\rho\overline{v^2}) + \nabla\cdot(\rho\overline{v^2}\boldsymbol{u}) - \nabla\cdot\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_k}\right)\nabla\overline{v^2}\right) = \rho k f - \rho\overline{v^2}\frac{\varepsilon}{k} \tag{2.9}$$

The equation (2.9) presents a turbulent production source term $\rho k f$ which models the redistribution of the turbulent kinetic energy by both the tangential (w.r.t to the streamlines) component and by the pressure echo effect. The destruction term of $\overline{v^2}$ is similar to the one of k as the dissipation is governed by the relaxation time k/ε).

The energy redistribution is modelled by the elliptic equation:

$$f - \lambda^2\boldsymbol{\Delta} f = C_2\frac{\mathbf{P}}{\rho k} - (1 - C_1)\frac{2/3 - \overline{v^2}/k}{T} \tag{2.10}$$

where the constants C_1 et C_2 are determined by DNS computations, and λ, which is homogeneous to k^3/ε^2, controls the characteristic diffusivity of the turbulent kinetic energy in the near-wall region. As described in [3], using asymptotic analysis, one can prove that f behaves as $-5\rho\overline{v^2}\varepsilon/k^2$ as the distance to the wall tends to zero . A change of variable is usually performed so that the boundary condition enforced at the wall becomes $f = 0$. The parameter S is introduced in the model such that the orginal formulation holds when $S = 1$ and the new formulation when $S = 6$.

Solving $\overline{v^2} - f$ system is performed as a step of the algorithm right after the $k - \varepsilon$ system is solved. The time-discrete $\overline{v^2} - f$ system reads:

$$\left|\begin{array}{l} \dfrac{\rho^n\overline{v^2}^{n+1} - \rho^{n-1}\overline{v^2}^n}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n\overline{v^2}^{n+1}\boldsymbol{u}^n) - \boldsymbol{\Delta}_{\mathcal{D},\mu_{\overline{v^2}}}(\overline{v^2}^{n+1}) = \rho^n k^{n+1} f^{n+1} - \rho^n S\dfrac{\overline{v^2}^{n+1}}{T(k^{n+1},\varepsilon^{n+1})} \\ f^{n+1} - \boldsymbol{\Delta}_{\mathcal{D}}^{\mu^{n+1}_{\overline{v^2}}}(\overline{v^2}^{n+1}) = C_1\dfrac{\mathbf{P}^n}{\rho^n k^{n+1}} + \dfrac{2}{3}\dfrac{(C_1-1)}{T(k^{n+1},\varepsilon^{n+1})} + \dfrac{(S-C_1)}{T(k^{n+1},\varepsilon^{n+1})}\dfrac{\overline{v^2}^{n+1}}{k^{n+1}} \end{array}\right.$$

where $\overline{v^2}$ represents a fraction of the turbulent kinetic energy which verifies the balance equation:

$$\frac{\rho^n k^{n+1} - \rho^{n-1}k^n}{\delta t} + \boldsymbol{\nabla}_{\mathcal{D}}\cdot(\rho^n k^{n+1}\boldsymbol{u}^n) - \boldsymbol{\Delta}_{\mathcal{D},\mu_{\overline{v^2}}}(k^{n+1}) = \mathbf{P}^n - \frac{\rho^n k^{n+1}}{T(k^n,\varepsilon^{n+1})}$$

If diffusion is neglected in the f–equation then:

$$\rho^n k^{n+1} f^{n+1} = C_2\mathbf{P}^n + \frac{2}{3}\frac{(C_1-1)}{T(k^{n+1},\varepsilon^{n+1})}\rho^n k^{n+1} - \frac{(S-C_1)}{T(k^{n+1},\varepsilon^{n+1})}\rho^n\overline{v^2}^{n+1}$$

If the time discretization of the of the turn over time T is the same in both k– and $\overline{v^2}$–equation, then by substracting balance equation of $\overline{v^2}$ to the two-thirds of the k–equation we get:

$$\mathcal{L}\left(\frac{2}{3}k^{n+1} - \overline{v^2}^{n+1}\right) + \frac{\rho^n C_1}{T(k^n,\varepsilon^{n+1})}\left(\frac{2}{3}k^{n+1} - \overline{v^2}^{n+1}\right) = \left(\frac{2}{3} - C_2\right)\mathbf{P}^n \tag{2.11}$$

where $\mathcal{L}$ is the convection–diffusion operator and the chosen discretization of the turn-over time is $T(k^n,\varepsilon^{n+1})$ which ensures the positivity of solution $(k^{n+1},\varepsilon^{n+1})$. The differential operator at the left-hand side verifies the M–matrix property and the source term is positive if $C_1 < 2/3$. Consequently for every $1 \leq n \leq N$, $\overline{v^2}^n$ is strictly bounded by $2/3k^n$.

2.6 Near wall treatment

One of the major difficulties of turbulence modeling to achieve computing efficiency with performance of the flow prediction lies in the treatment of walls. The evaluation of turbulent scales in the near-wall region is crucial to ensure a valid solution with regard to the physics since the turbulent shear stresses are dominant in the vicinity of the wall.

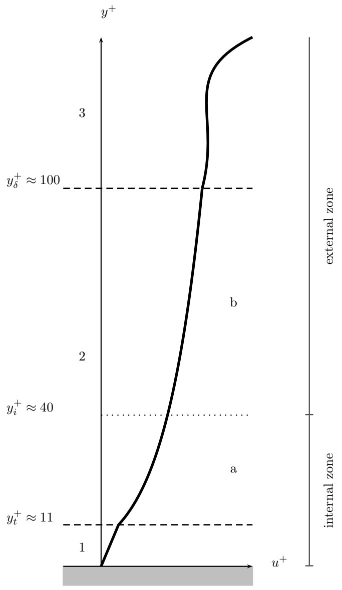
In the case of the $k - \varepsilon$ model, the boundary conditions for the turbulent scales enforced on walls by the physics are, for any $\boldsymbol{x} \in \partial\Omega$:

1. $k(\boldsymbol{x}) = 0$, $\boldsymbol{\nabla}k(\boldsymbol{x})\cdot\boldsymbol{n} = 0$
2. $\varepsilon(\boldsymbol{x}) > 0$, $\boldsymbol{\nabla}\varepsilon(\boldsymbol{x})\cdot\boldsymbol{n} = 0$

where $\boldsymbol{n}$ denotes the unit vector outward to the wall. On the one hand, computing accurately the near-wall solution while enforcing these boundary conditions would imply a meshing of the order of the turbulent lentgh scale, which amount to the Kolmogorov scale as the distance to the wall tends to zero. On the other hand the condition on ε makes the problem ill-posed.

Turbulent boundary layers are usually modeled after the multilayer phenomenological approach depicted in Figure 2.1 which allows to express turbulent scales as functions of the nondimensionalized velocity u^+. The obtained velocity profile is often called “universal profile” since its expression follows from dimensional analysis in the infinit Reyonlds limit.

Wall-laws are usually obtained by projecting the balance equations in the plane normal to the wall and resolving ordinary differential equations derived from asymptotic analysis as the distance to the wall tends to zero. The turbulent scales are thus evaluated as functions of the friction velocity u_τ which takes into account of the turbulent shear in the vicinity of the wall.



1. viscous sublayer: in this zone corresponding to $y^+ \in [0, y_t^+]$ viscous effects are overwhelming, the profile of the nondimensionalized velocity is linear.
2. logarithmic sublayer: in this zone composed of two sublayers, the buffer sublayer (a) and the inertial sublayer (b) turbulent effects are of the order of viscous effect and turbulent production balances dissipation.
3. central zone: turbulent scales are governed by their respective balance equations.

Figure 2.1: Turbulent boundary layer: multilayered approach and taxinomy.

2.6.1 Elements of dimensional analysis

We recall how wall functions are evaluated for the treatment of turbulence for wall-bounded flows, which is described for instance in [8]. Governing equations are studied in a steady boundary layer in the case of the flow in a plane duct with zero pressure gradient. In the near-wall region the strain stress reads $\tau = \rho\nu_\ell \boldsymbol{\nabla}\bar{u}\cdot\boldsymbol{n} - \rho\overline{u'v'}$, where n denotes the outward normal to the wall, ν_ℓ stands for the kinematic viscosity, $\mu_\ell = \rho\nu_\ell$ denotes the dynamic viscosity and u, v denote respectively the tangential and normal component of the velocity. The friction velocity is defined as a function of the wall stress τ_w by relation:

$$\tau_w = \rho\nu_\ell \boldsymbol{\nabla}\bar{u}\cdot\boldsymbol{n} = \rho(u_\tau)^2$$

since turbulence stresses are negligible. Then,

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \tag{2.12}$$

The nondimensionalized length scale and time scale are:

$$u^+ = \frac{\bar{u}}{u_\tau} \qquad y^+ = \frac{y u_\tau}{\nu_\ell} \qquad {\nu_t}^+ = \frac{\nu_t}{\nu_\ell}$$

In the developed region of the plane duct, the momentum equation reduces to the diffusion term since transport vanishes with steadiness assumption and invariance w.r.t space variables x and z, then:

$$\frac{\partial}{\partial y}(\mu_\ell + \mu_t)\frac{\partial u}{\partial y} = 0$$

By integrating over a fictitious fluid volume:

$$\int_V \frac{\partial}{\partial y}(\mu_\ell + \mu_t)\frac{\partial u}{\partial y}\mathrm{d}y = \int_{\partial V}(\mu_\ell + \mu_t)\frac{\partial u}{\partial y}\cdot\boldsymbol{n}\mathrm{d}s = |\partial\bar{v}|\tau^+$$

with the definition of friction velocity u_τ, equation of the y–component reads :

$$(\mu_\ell + \mu_t)\frac{\partial u}{\partial y} = \rho(u_\tau)^2$$

The nondimensionalized relation w.r.t u^+ and y^+ reads:

$$(1 + {\nu_t}^+)\frac{\mathrm{d}u^+}{\mathrm{d}y^+} = 1$$

In the viscous sublayer turbulent viscosity is negligible, ${\nu_t}^+ << 1$, then :

$$\frac{\mathrm{d}u^+}{\mathrm{d}y^+} = 1 \Rightarrow u^+ = y^+$$

and in the logarithmic layer, ${\nu_t}^+ >> 1$:

$${\nu_t}^+ = \frac{\mathrm{d}y^+}{\mathrm{d}u^+}$$

Using the Prandtl relation, we have :

$$\mu_t = \rho C_\mu \sqrt{k}\ell \Rightarrow {\nu_t}^+ = \kappa y^+$$

then

$$\frac{1}{\kappa}\frac{y^+}{\mathrm{d}y^+} = \mathrm{d}u^+ \Rightarrow u^+ = \frac{1}{\kappa}\ln(y^+) + E^*$$

The near-wall profile of turbulent viscosity is given by relation:

$${\nu_t}^+ = \frac{\mathrm{d}u^+}{\mathrm{d}y^+} - 1$$

As $u^+ = y^+$ in the viscous sublayer, then ${\nu_t}^+ = 0$ which is consistent with the assumption that the turbulent viscosity can been neglected, and in the log-layer ${\nu_t}^+$ is a linear function and proportional to the Von Kármán constant κ. If the turbulent viscosity is evaluated by Prandtl-Kolmogorov relation $\nu_t = C_\mu k^2/\varepsilon$ then the turbulent kinetic energy k^+ decreases as a quadratic function of y^+. Similarily, the profiles of any turbulent scales can be evaluated in the log-layer and viscous layer by solving an symptotic system obtained by simplifying of the model equations with the "right" hypotheses.

Moreover, dimensional analysis can be performed to evaluate the pertinence of the model constant values obtained from DNS experiments. For instance we can remark that the model constant C_μ is considerably lower in the $\overline{v^2} - f$ model than in the $k - \varepsilon$ models. This fact can be assessed using the following dimensional analysis. The Boussinesq hypothesis is enforced so that the turbulent viscosity is defined thanks to a mean gradient approximation:

$$-\rho\overline{\boldsymbol{u'} \otimes \boldsymbol{u'}} : \boldsymbol{\nabla}\bar{\boldsymbol{u}} = \mu_t \frac{1}{2}(\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{\nabla}^t\bar{\boldsymbol{u}}) : \boldsymbol{\nabla}\bar{\boldsymbol{u}}$$

since $\mu_t = -\dfrac{\overline{u'v'}}{\boldsymbol{\nabla}\bar{\boldsymbol{u}} \cdot \boldsymbol{n}}$

Finally, again with $\boldsymbol{u'} = (u', v')^t$ the fluctuating velocity field, $\bar{u}$ the tangential component of the mean velocity field and $\boldsymbol{n}$ is the outward normal to the wall, the balance equation for k reduces to:

$$-\overline{u'v'}\boldsymbol{\nabla}\bar{u} \cdot \boldsymbol{n} = \varepsilon$$

Using the Prandtl-Kolmogorov hypothesis (2.3), we get:

$$C_\mu = -\frac{\overline{u'v'}\boldsymbol{\nabla}\bar{u} \cdot \boldsymbol{n}}{k^2}\frac{\mu_t}{\rho}$$

soit

$$C_\mu = \frac{(\overline{u'v'})^2}{k^2}$$

The constant C_μ can be interpreted as ratio accounting for the anisotropy in the boundary layer. An optimal value of C_μ can be determined to ensure the proper decay of the turbulent viscosity in the near-wall limit. Thus, the difference between the values of C_μ in the $k - \varepsilon$ and $\overline{v^2} - f$ models can be interpreted as a consequence of the isotropy hypothesis made in the case of the standard $k - \varepsilon$ model.

2.6.2 $k - \varepsilon$ model

The evaluation of the turbulent scales in the case of the off-wall strategy implemented in ISIS, is achieved using the following nondimensionalized scales:

$$k^+ = \frac{|u_\tau|^2}{\sqrt{C_\mu}} \quad , \quad \varepsilon^+ = \frac{|u_\tau|^3}{\kappa\delta} \tag{2.13}$$

with δ the turbulent boundary layer thickness and κ the Karman constant. According to the previous asymptotic analysis yields the following relations to compute the velocity scale in the different layers of the near-wall region:

$$u^+ = \begin{cases} y^+ & \text{if } y^+ < y_t^+ \\ \frac{1}{\kappa}\ln(Ey^+) & \text{if } y^+ > y_t^+ \end{cases} \tag{2.14}$$

where the value of the Karman constant is $\kappa = 0.419$, and constant $E = 9.793$ is solution to $y_t^+ = 1/\kappa \ln(Ey_t^+)$ for a given the nondimensionalized distance of the transition to the log-layer y_t^+, which is usually deduced from DNS computations.

2.6.3 $k - \varepsilon - \overline{v^2} - f$ model

Following the ideas from [3], we describe how wall-laws can be derived for the $k - \varepsilon - \overline{v^2} - f$ model from the asymptotic balance equations governing the evolution of the nondimensionalized turbulent scales in the wall vicinity and using some arguments of dimensional analysis. We reevaluate the constants and the functions from the original paper to ensure continuity of the profiles between the viscous sublayer and the logarithmic layer.

In the steady boundary layer of a wall-bounded shear flow, since variables do not depend on time t and space variables x and z, the equations of the $k - \varepsilon - \overline{v^2} - f$ model reduce to:

$$-\frac{\partial}{\partial y}\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right) = \mathbf{P} - \rho\varepsilon$$

$$-\frac{\partial}{\partial y}\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right) = \frac{1}{T}(C_{\varepsilon 1}^z\mathbf{P} - \rho C_{\varepsilon 2}\varepsilon)$$

$$-\frac{\partial}{\partial y}\left(\left(\mu_\ell + \frac{\mu_t}{\sigma_k}\right)\frac{\partial \overline{v^2}}{\partial y}\right) = \rho k f - S\rho\overline{v^2}\frac{\varepsilon}{k}$$

$$f - \frac{\partial}{\partial y}\left(\lambda^2\frac{\partial f}{\partial y}\right) = C_2\frac{\mathbf{P}}{\rho k} + (C_1 - 1)\frac{2/3}{T} + (S - C_1)\frac{\overline{v^2}/k}{T}$$

where y is the normal component to the wall.

For each turbulent variable χ, the corresponding nondimensionalized turbulent scale is denoted by χ^+:

$$k^+ = \frac{k}{u_\tau^2} \qquad \varepsilon^+ = \frac{\varepsilon\nu_\ell}{u_\tau^4} \qquad \overline{v^2}^+ = \frac{\overline{v^2}}{u_\tau^2} \qquad f^+ = \frac{f\nu_\ell}{u_\tau^2} \tag{2.15}$$

2.6.3.1 Analytical solutions in the viscous sublayer

In the viscous sublayer, the turbulent stresses remain negligible with respect to viscous stresses; consequently both turbulent diffusion and turbulent production are neglected:

$$
\begin{aligned}
-\frac{\partial}{\partial y}\left(\mu_\ell \frac{\partial k}{\partial y}\right) &= -\rho\varepsilon \\
-\frac{\partial}{\partial y}\left(\mu_\ell \frac{\partial \varepsilon}{\partial y}\right) &= -\frac{1}{T}\rho C_{\varepsilon 2}\varepsilon \\
-\frac{\partial}{\partial y}\left(\mu_\ell \frac{\partial \overline{v^2}}{\partial y}\right) &= \rho k f - S\rho\overline{v^2}\frac{\varepsilon}{k} \\
f - \frac{\partial}{\partial y}\left(\lambda^2 \frac{\partial f}{\partial y}\right) &= (C_1 - 1)\frac{2/3}{T} + (S - C_1)\frac{\overline{v^2}/k}{T}
\end{aligned}
$$

Nondimensionalized turbulent scales are solution to the a system of ordinary differential equations:

$$
\left|
\begin{aligned}
-(k^+)'' + \varepsilon^+ &= 0 \\
-(\varepsilon^+)'' + C_{\varepsilon 2}\frac{(\varepsilon^+)^2}{k^+} &= 0 \\
-(\overline{v^2}^+)' &= k^+ f^+ - S\overline{v^2}^+\frac{\varepsilon^+}{k^+} \\
f^+ - \lambda^2 \frac{u_\tau^2}{\nu_\ell}(f^+)'' &= (C_1 - 1)\frac{2/3}{T^+} + (S - C_1)\frac{\overline{v^2}^+/k^+}{T^+}
\end{aligned}
\right.
\tag{2.16}
$$

where $(\chi^+)'$ is the derivative of the nondimensionalized variable associated to χ with respect to the nondimensionalized length scale y^+.

The k– and ε–equations can be uncoupled thanks to the assumption that the turnover time scale $T = k/\varepsilon$ is equal to the Kolmogorov time scale τ_K in the viscous sublayer. Then the ε–equation reads:

$$
-(\varepsilon^+)'' + \frac{C_{\varepsilon 2}}{\tau_K^+}(\varepsilon^+)^2 = 0
$$

where $\tau_K^+ = C_\tau/\sqrt{\varepsilon^+}$ is the nondimensionalized Kolmogorov time scale and the value $C_\tau = 6$ is deduced from DNS computations. This assumption is consistent with the fact that the characteristic length of eddies is of the same order as the dissipation scale in regions where viscous effects are overwhelming. A trivial solution to this differential equation is $\varepsilon^+ \sim (y^+ + C)^{-4}$ with C an integration constant to be determined in the latter thanks to the boundary conditions. Let be A a constant such that:

$$
\varepsilon^+ = \frac{A}{C_{\varepsilon 2}^2}(y^+ + C)^{-4}
$$

then constant A satisfies $A/C_\tau - 20A^{3/2} = 0$.

Using the k–equation the expression of k^+ can be devised by integrating ε^+ two times w.r.t y^+: $k^+ \sim B_1(y^+ + C)^{-2} + B_2 y^+ + B_3$. By homogeneity, constant B_1 is equal to $A/6C_{\varepsilon 2}^2$ while constants B_2 and B_3 are respectively determined by enforcing the boundary conditions $(k^+)'(0) = 0$ and $k^+(0) = 0$:

$$
\begin{aligned}
(k^+)'(y^+) &= \frac{A}{3C_{\varepsilon 2}^2}(y^+ + C)^{-3} + B_2 && \Leftrightarrow B_2 = 2B_1/C^3 \\
k^+(y^+) &= \frac{A}{6C_{\varepsilon 2}^2}(y^+ + C)^{-2} + B_2 y^+ + B_3 && \Leftrightarrow B_3 = -B_1/C^2
\end{aligned}
$$

The profile $\overline{v^2}^+ \sim (y^+)^4$ is enforced, while the expression of f^+ is obtained by solving the fourth equation of the system, assuming that last term can be neglected as $y^+ \to 0$ (the ratio $\overline{v^2}^+/k$ vanishes while $T^+ \to \tau_K$).

2.6.3.2 Analytical solutions in the log-layer

In the log-layer the turbulent production and the turbulent diffusion are supposed to be dominant w.r.t the molecular diffusion, then the equations for the length scales read:

$$\begin{aligned} -\frac{\partial}{\partial y}\left(\nu_t \frac{\partial k}{\partial y}\right) + \nu_t \left|\frac{\partial u}{\partial y}\right|^2 - \varepsilon &= 0 \\ -\frac{\partial}{\partial y}\left(\nu_t \frac{\partial \overline{v^2}}{\partial y}\right) + kf - S\frac{\overline{v^2}}{k}\varepsilon &= 0 \end{aligned}$$

Moreover the turbulent production and the dissipation are supposed to cancel with each other:

$$\nu_t^+ \left|\frac{\partial u^+}{\partial y}\right|^2 - \varepsilon^+ = 0 \tag{2.17}$$

then the first equation reduces to:

$$-\frac{\partial}{\partial y^+}\left(\nu_t^+ \frac{\partial k}{\partial y^+} k^+\right) = 0$$

then we conclude that $\nu_t^+ = \kappa y^+$ similarily to the $k - \varepsilon$ case.

The nondimensionalized profiles for k^+ and $\overline{v^2}^+$ in the log-layer are given by:

$$k^+ = \frac{C_k}{\kappa}\ln(y^+) + B_k \qquad \text{and} \qquad \overline{v^2}^+ = \frac{C_{\overline{v^2}}}{\kappa}\ln(y^+) + B_{\overline{v^2}}$$

if we suppose that similarly to k^+, that the condition of balance between production and dissipation is satisfied:

$$k^+ f^+ - S\overline{v^2}^+ / k^+ \varepsilon^+ = 0 \tag{2.18}$$

and the values of the constants B_k, $B_{\overline{v^2}}$, C_k and $C_{\overline{v^2}}$ are deduced from DNS computations.

Using the hypothesis 2.17, we deduce that:

$$\varepsilon^+ = \frac{1}{\nu_t^+} = \frac{1}{\kappa y^+}$$

since $\partial u^+/\partial y^+ = 1/\nu_t^+ \ \nu_t^+ \partial u/\partial y = u_\tau$ in the log-layer.

The equation governing the profile of f^+ is deduced from (2.18):

$$f^+ \sim \frac{S\overline{v^2}^+}{(k^+)^2}\varepsilon^+$$

and modified so that the profile vanishes as $y^+ \to \infty$ and to ensure the continuity of the function at $y^+ = y_t^+$ by setting its value to:

$$f_t^+ = \frac{S\overline{v^2}_t^+ \varepsilon_t^+}{(k_t^+)^2 + \exp(\varepsilon_t^+ / k_t^+)}$$

with $a_t^+ = a(y_t^+)$, the value computed at the transition.

2.6.3.3 Profiles

The profiles of the turbulent scales are gathered here as well as the constants in the log-layer in Table 2.2.

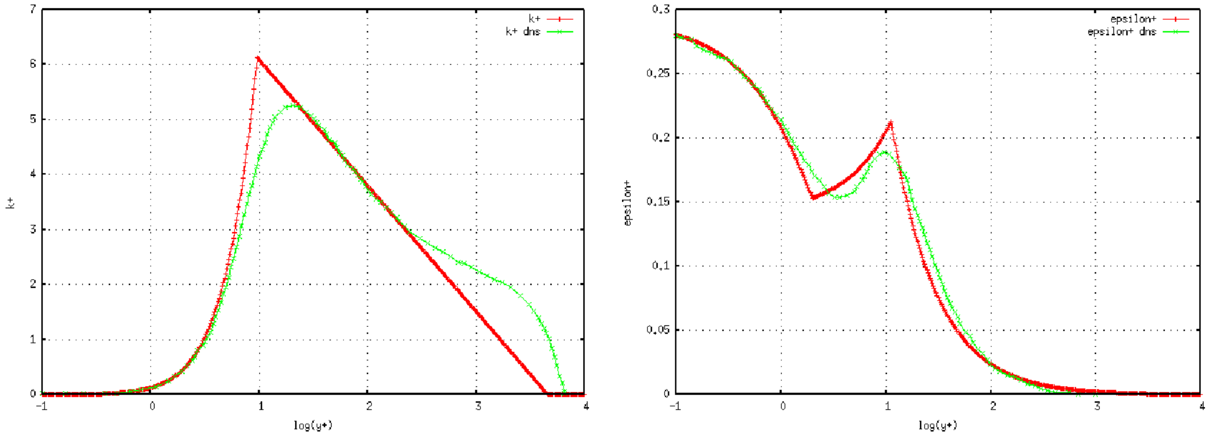


Figure 2.2: Comparison of the wall-law profiles of k and ϵ (red) with the profiles from a DNS of Spalart [9] (green), as a function of y^+

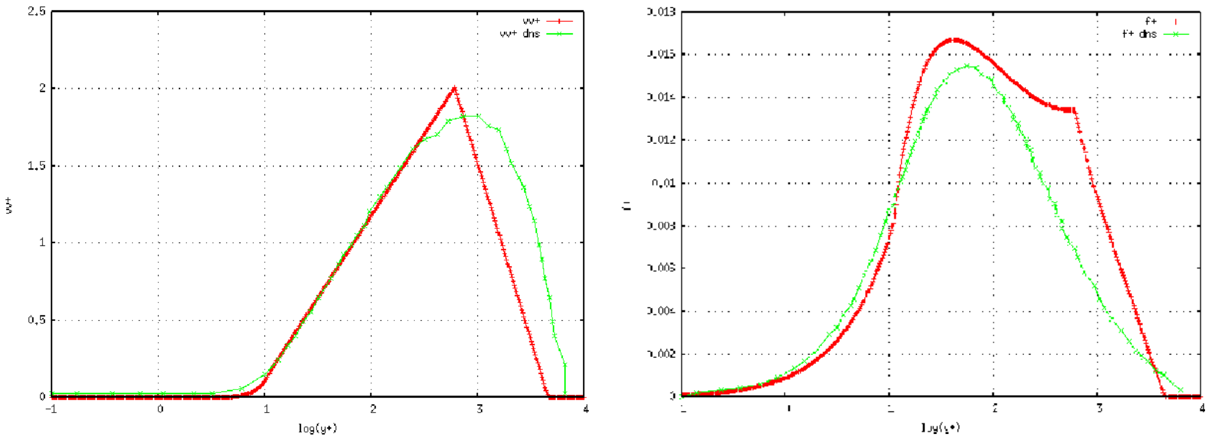


Figure 2.3: Comparison of the wall-law profiles of $\overline{v^2}$ and f (red) with the profiles from a DNS of of Spalart [9] (green), as a function of y^+

Viscous sublayer	Log-layer
$k_v^+ = \dfrac{A}{6C_{\varepsilon 2}^2}\left(\dfrac{1}{(y^+ + B)^2} + \dfrac{2y^+}{B^3} - \dfrac{1}{B^2}\right)$	$k_l^+ = \dfrac{C_k}{\kappa}\ln(y^+) + B_k$
$\varepsilon_v^+ = \dfrac{A}{C_{\varepsilon 2}^2}\dfrac{1}{(y^+ + B)^4}$	$\varepsilon_l^+ = \dfrac{1}{\kappa y^+}$
$\overline{v^2}_v^+ = K_{\overline{v^2}} C_{\overline{v^2}} (y^+)^4$	$\overline{v^2}_l^+ = \dfrac{C_{\overline{v^2}}}{\kappa}\ln(y^+) + B_{\overline{v^2}}$
$f_v^+ = C_1 (y^+ + B)^{0.5+\sqrt{D}} + C_2 (y^+ + B)^{0.5-\sqrt{D}} - \dfrac{C_3}{(y^+ + B)^2}$	$f_l^+ = K_f \dfrac{S\overline{v^2}^+ \varepsilon^+}{(k^+)^2 + \exp(\varepsilon^+/k^+)}$

C_3	D	$K_{\overline{v^2}}$	K_f
$B^2 (C_1 B^{0.5+\sqrt{D}} + C_2 B^{0.5-\sqrt{D}})$	$\dfrac{1}{4} + \dfrac{120}{C_{\varepsilon 2} C_\eta^2 C_L^2}$	$\dfrac{\ln(y_t^+)}{(\kappa + B_{\overline{v^2}}/C_{\overline{v^2}})(y_t^+)^4}$	$\dfrac{f_v^+(y_t^+)}{f_t^+}$

Table 2.2: Analytical solutions for the nondimensionalized turbulent variables

A	B	C_1	C_2
14400	11.516	5.05×10^{-4}	4.95×10^{-3}

C_k	B_k	$C_{\overline{v^2}}$	$B_{\overline{v^2}}$
−0.416	8.366	0.193	−0.940

Table 2.3: Model constants in the viscous sublayer with $C_\tau = 6$ and in the log-layer

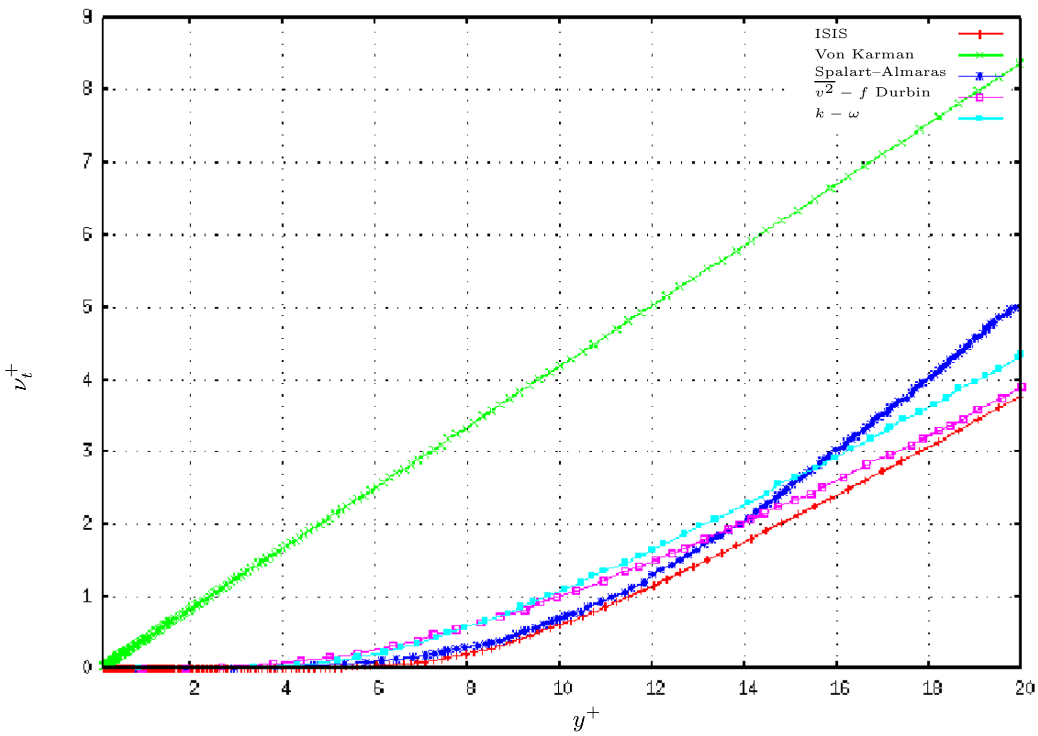


Figure 2.4: Comparison of the turbulent viscosity ν_t^+ profiles as a function of y^+ at the vicinity of the wall: implemented version (red), Von Karman law (green), Spalart–Almaras (blue), $\overline{v^2} - f$ Durbin [3] (magenta), $k - \omega$ (cyan)

2.7 Numerical tests

In this section we perform a convergence analysis of the numerical scheme given by (2.4) by means of the method of manufactured solutions. Given any smooth analytical field $\tilde{s} = (\tilde{k}(t,x), \tilde{\varepsilon}(t,x))^T$ defined on a bounded domain Ω and over a time-interval $[0,T]$, let assume that $\tilde{s}$ is solution of the continuous problem $\mathcal{A}\,\tilde{s} = b(\tilde{s})$, where $\mathcal{A}$ is a given differential operator an b a real function. Then s is a discrete solution of the system $\mathcal{A}_\mathcal{D}\, s = b(s) + q_\mathcal{D}(\tilde{s})$, where $q_\mathcal{D}(\tilde{s}) = \mathbf{P}_\mathcal{M}[\mathcal{A}(\,\tilde{s}) - b(\tilde{s})]$ is a source term build by projecting $\mathcal{A}\,\tilde{s} - b(\tilde{s})$ on the discrete space, for instance defined as follows:

$$\text{for all } x \in \Omega \text{ such that } x \in K\,,\ \mathbf{P}_\mathcal{M}\tilde{s}(\boldsymbol{x}) = \tilde{s}(x_K)$$

where K is a control volume of mesh $\mathcal{M}$ and x_K its centre.

Both time-independent and -dependent cases are addressed. The computational domain is $(0,1)\times(0,1)$ and meshes used in this study are 20×20, 40×40, 80×80, 160×160 regular grids. In addition to the implicit upwind approximation for the convection fluxes theoretically studied above, an explicit MUSCL discretization is implemented. In both the steady and unsteady case, the behaviour of the upwind and the MUSCL schemes is assessed. We choose $u(\boldsymbol{x}) = (1+2x_2x_1^2, -1-2x_1x_2^2)$ for the (steady and divergence-free) velocity field and the following analytical solution:

$$k(t,x) = 1 + f(t)\frac{\cos(\pi x_1) - \sin(\pi x_2)}{4}, \quad \varepsilon(t,x) = 1 + g(t)\frac{\cos(\pi x_1) + \sin(\pi x_2)}{4}$$

with $f(t) = g(t) = 1$ in the steady case and $f(t) = \sin(\pi t)$, $g(t) = \sin(\pi t/2)$ in the unsteady case. The finite-volume L^2 error norm used in the following is the usual discrete norm:

$$\|s - \tilde{s}\|_{\Omega,\mathcal{D}} = \left[\sum_{K\in\mathcal{M}} m(K)(s_K - s(x_K))^2\right]^{1/2}$$

Non-linear system (2.4) is solved by means of a Newton method, and no evidence of possible multiple solutions is observed.

Results are reported in Figure 2.5 for the steady case. They show the expected near first-order in space behaviour for the upwind discretization. For the MUSCL algorithm, the convergence rate lies between 1.5 and 2, and tends to 2 when refining the meshing.

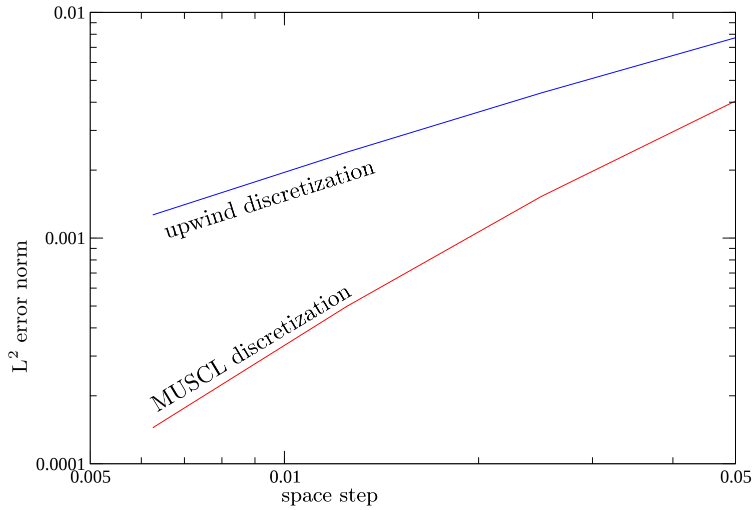


Figure 2.5: Steady case – Discrete L^2 norm of the error for the upwind discretization and for the MUSCL discretization.

In the unsteady case and for the upwind scheme, first-order convergence with respect to the space and time-step at constant CFL number is assessed in Figure 2.6. For the MUSCL discretization, the curves of Figure 2.7 show a decrease with the time step for large values of this discretization parameter; then, a plateau is reached, due to the residual error in space. The order of convergence with respect to time is still one.

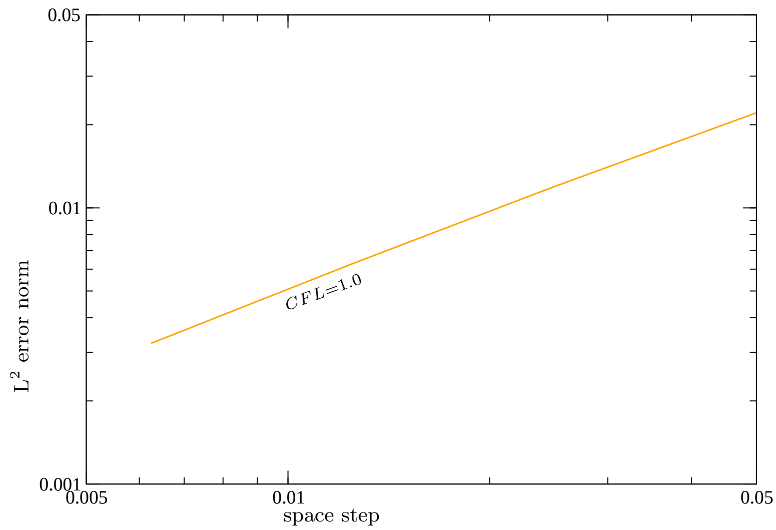


Figure 2.6: Unsteady case – Discrete L^2 norm of the error as a function of the space step with constant Courant number (CFL=1.0) for the upwind discretization.

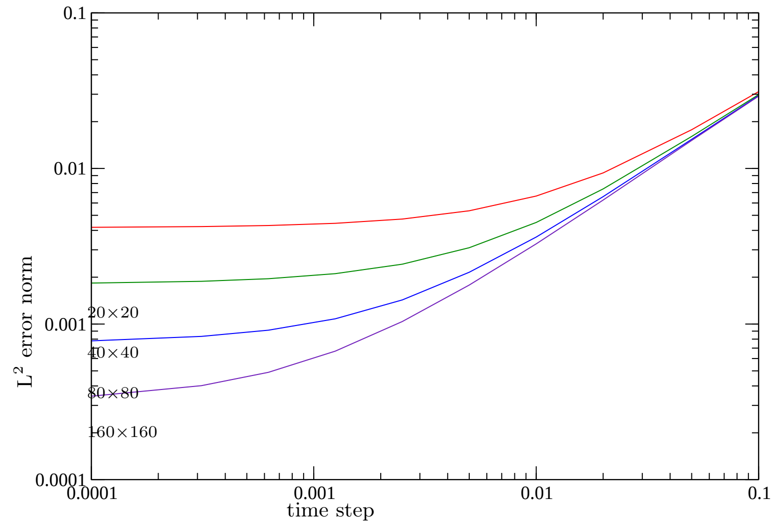


Figure 2.7: Unsteady case – Discrete L^2 norm of the error as a function of the time step, for the MUSCL discretization.

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Chapitre 3

Convergence analysis of a finite element/finite volume scheme for a RANS turbulence model

CONVERGENCE ANALYSIS OF A FINITE ELEMENT/FINITE VOLUME SCHEME FOR A RANS TURBULENCE MODEL

Abstract.In this paper, we study the stability and convergence of a numerical approximation of a nonlinear system of elliptic equations encountered in Reynolds Averaged Navier-Stokes turbulence models. The addressed problem consists in the coupling of the steady Stokes equations discretized by a nonconforming finite element technique with a convection-diffusion equation for a turbulent energy discretized with a finite volume method. The right-hand side of the latter is the viscous dissipation associated to the Stokes equations, and thus couples the system and only lies in L^1. We first prove (independent of the mesh step) *a priori* estimates satisfied by any discrete solution. These estimates yield a compactness result for a sequence of solutions obtained with meshes the step of which tends to zero. Up to the extraction of a subsequence, we thus obtain the existence of a limit, which is shown to be the solution to a weak formulation of the problem.

3.1 Introduction

The addressed problem is the coupling of the steady incompressible Stokes equations with a convection–diffusion equation for a turbulent scale denoted by k:

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\big(\lambda(k)\boldsymbol{\nabla}\boldsymbol{u}\big)+\boldsymbol{\nabla}p=\boldsymbol{f}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \end{aligned} \tag{3.1a}$$

$$-\boldsymbol{\nabla}\cdot\big(\lambda(k)\boldsymbol{\nabla}k\big)+\boldsymbol{\nabla}\cdot\big(k\boldsymbol{u}\big)=\lambda(k)|\boldsymbol{\nabla}\boldsymbol{u}|^2, \tag{3.1b}$$

$$\boldsymbol{u}=0\ ,\ k=0 \qquad \text{on } \partial\Omega, \tag{3.1c}$$

where $\boldsymbol{u}$ and p stand respectively for the velocity field and the pressure, and $\boldsymbol{f}$ is a forcing term. This system of equations is posed over Ω, an open bounded set of $\mathbb{R}^d$, $d=2,3$, supposed to be polygonal ($d=2$) or polyhedral ($d=3$). For the sake of simplicity, only homogeneous Dirichlet boundary conditions are considered on $\partial\Omega$ for the velocity $\boldsymbol{u}$ and the turbulent scale k, and the forcing term $\boldsymbol{f}$ is supposed to be bounded in $L^2(\Omega)^d$.

The coefficient λ, called *effective viscosity* in the frame of eddy viscosity models, results from the combination of the molecular viscosity μ and an additional diffusivity μ_t called *turbulent viscosity* which takes into account the turbulent stresses. In this model problem, μ is supposed to remain constant and positive, while μ_t depends on the turbulent scale k. The chosen algebraic relation is a law similar to the well-known one-equation Prandtl model:

$$\lambda(k)=\min\left[\lambda_\infty,\ (\mu^2+\ell^2k)^{1/2}\right] \tag{3.2}$$

where λ_∞ is a positive real number satisfying $\lambda_\infty>\mu$, and ℓ is a non-negative real number.

Problem (3.1) can be considered as a reduced turbulence model which retains some of the main mathematical difficulties posed by the analysis of Reynolds Averaged Navier–Stokes (RANS) turbulence models. Indeed, as in usual physical models, the incompressible Stokes problem is coupled to the turbulent transport equation (3.1b) in two ways: first by the nonlinear diffusion coefficient $\lambda(k)$ and, second, by the turbulent production term at the right-hand side of (3.1b). For this latter, the classical energy estimates applied to the Stokes equations (3.1a) yields that $\lambda(k)|\boldsymbol{\nabla}\boldsymbol{u}|^2$ belongs to $L^1(\Omega)$, so the right-hand side of the scalar convection–diffusion equation (3.1b) for k only belongs to $L^1(\Omega)$.

Stokes equations 3.1a are discretized by a nonconforming finite element technique while the convection–diffusion equation (3.1b) is approximated by a standard finite volume method. The choice of this hybrid discretization is motivated by the fact that the built-in divergence conservation property of the Crouzeix–Raviart element and an upwind discretization of the balance equation for the turbulence scale k yields a scheme which ensures the positivity of k, which is coherent with the physics since this quantity is usually homogeneous to an energy (more precisely speaking, k is often identified to the turbulent energy, *i.e.* the kinetic energy associated to the fluctuating part of the velocity).

The aim of this paper is to prove the convergence of the proposed finite element/finite volume scheme in the sense that any sequence of solutions $(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)})_{m\in\mathbb{N}}$, obtained with a sequence of meshes the step of which tends to zero, converges (up to a subsequence) to a function $(\boldsymbol{u}, p, k)$ which is solution to the problem in the following weak sense:

$$
\begin{aligned}
&(\boldsymbol{u}, p, k) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega) \times \mathrm{W}_0^{1,\alpha}(\Omega), \text{ for any } \alpha \in [1, d/(d-1)) \text{ and,}\\
&\text{for all } (\boldsymbol{v}, q, \psi) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{C}_c^\infty(\Omega) :\\
&\qquad \int_\Omega \lambda(k)\ \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \boldsymbol{\nabla}\cdot\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\
&\qquad \int_\Omega q\ \boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = 0,\\
&\qquad \int_\Omega \lambda(k)\ \boldsymbol{\nabla}k\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} - \int_\Omega k\ \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k)\ |\boldsymbol{\nabla}\boldsymbol{u}|^2\ \psi\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{3.3}
$$

where $\mathrm{L}_0^2(\Omega)$ stands for the space of functions of $\mathrm{L}^2(\Omega)$ with zero mean value.

The literature related to this work addresses the mathematical analysis, at the continuous level, and the convergence of numerical schemes for two kind of problems of increasing complexity: first, single elliptic or parabolic equations with L^1 (or measure) data; second, problems coupling two elliptic equations, the right-hand side of the second one, as here, being the energy associated to the first one. The mathematical analysis of a class of nonlinear elliptic and parabolic problems with irregular data has first been addressed in [3], and further developments can be found in [2]. In this frame, existence of a weak solution belonging to some Sobolev space $\mathrm{W}_0^{1,\alpha}(\Omega)$, where α satisfies a condition on the critical exponent of a Sobolev embedding has been proven for the so-called p–Laplacian. The equivalent discrete *a priori* estimates, and then the convergence of the numerical scheme, have been obtained in [11] in the case of a finite volume discretization of the Laplace problem. Turning now to the study of systems of the aforementioned structure, the existence results of [3] have been partly extended in [6], this work being motivated by the modelling of an induction heating problem; the convergence of a Lagrange finite element discretization is then shown in the same work. Then, finite volume et finite element techniques have been investigated in [4] for the discretization of a similar physical model. Existence of a solution to System (3.3) may be found in [17], both for the (Navier-Stokes) steady and (Stokes in 2D or 3D and Navier-Stokes in 2D) unsteady cases. To our knowledge, the convergence result presented in the present paper is the first one for the discretization of (3.3).

The presentation is organized as follows: we first introduce in Section 3.2, the (finite element and finite volume) discretization, together with some useful properties of the considered approximation spaces. Then we describe the numerical scheme (Section 3.3) and establish *a priori* estimates (Section 3.4). Finally, we show in Section 3.5 that these estimates yield the relative compactness of a family of approximate solutions to the discrete problem, and that any limit solves the continuous problem.

3.2 Discrete functional framework and numerical scheme

3.2.1 The mesh

Let $\mathcal{M}$ be a partition of the domain Ω into disjoints simplices, supposed to be regular in the usual sense of the finite element literature; in particular for any elements $K, L \in \mathcal{M}$, $\bar{K} \cap \bar{L}$ is either reduced to $\emptyset$, a vertex, (for $d = 3$) a segment, or a whole face. The set of the faces of the mesh is denoted by $\mathcal{E}$, $\mathcal{E}_{\text{ext}}$ stands for the set of faces included in the boundary of Ω while the set of internal faces $\mathcal{E} \setminus \mathcal{E}_{\text{ext}}$ is denoted by $\mathcal{E}_{\text{int}}$. For every element $K \in \mathcal{M}$, $\mathcal{E}(K)$ represents the set of faces of K. The internal face $\sigma \in \mathcal{E}_{\text{int}}$ separating the control volumes K and L is denoted by $\sigma = K|L$. By $|\cdot|$ we denote either the d–dimensional or $(d-1)$–dimensional Lebesgue measure such that $|K|$ and $|\sigma|$ represent respectively the measure of the element K and the face σ.

Moreover, for the discretization of a diffusion term by the finite volume method, we suppose that there exists a family $\mathcal{P}$ such that $x_K \in \bar{K}$ for all $K \in \mathcal{M}$ and, for any internal face $\sigma = K|L$, $x_K \neq x_L$ and the straight line going through x_K and x_L is orthogonal to σ. For any control volume K and face σ of K, we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ and by $\boldsymbol{n}_{K,\sigma}$ the unit vector normal to σ outward from K. For any edge or face σ, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, if σ separates the two control volumes K and L (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if σ is included in the boundary.

We measure the regularity of the mesh by the parameter $\theta_{\mathcal{M}} > 0$ defined by:

$$\theta_{\mathcal{M}} = \inf_{K\in\mathcal{M}} \left\{ \frac{d_{K,\sigma}}{d_\sigma}; \sigma \in \mathcal{E}(K) \right\} \cup \left\{ \frac{d_{K,\sigma}}{\operatorname{diam}(K)}; \sigma \in \mathcal{E}(K) \right\} \tag{3.4}$$

The size of the mesh, $h_{\mathcal{M}}$, is defined by:

$$h_{\mathcal{M}} = \sup_{K\in\mathcal{M}} \operatorname{diam}(K).$$

Definition 3.2.1 (Regular sequence of discretizations)**.**
A sequence $\left(\mathcal{M}^{(m)}\right)_{m\in\mathbb{N}}$ of Ω is said regular if:

1. $h_{\mathcal{M}}^{(m)} \to 0$ as $m \to \infty$,
2. there exists $\theta_0 > 0$ such that $\theta_{\mathcal{M}}^{(m)} \geq \theta_0$, $\forall m \in \mathbb{N}$, with $\theta_{\mathcal{M}}^{(m)}$ defined by (3.4).

3.2.2 Nonconforming finite elements

For the (lowest order) Crouzeix–Raviart element, the discrete space for each component of the velocity is included in the space of piecewise affine polynomials. The mean value of the jump across an internal edge of any discrete function is required to vanish, which gives sense to the following set of functionals:

$$\mathrm{F}_\sigma(v) = \frac{1}{|\sigma|} \int_\sigma v(\boldsymbol{x})\,\mathrm{d}\gamma, \qquad \forall \sigma \in \mathcal{E}, \tag{3.5}$$

where $\mathrm{d}\gamma$ denotes the $(d-1)$-dimensional Lebesgue measure. As usual in the finite element framework, the Dirichlet boundary conditions are enforced by the choice of the approximation space, so the functional space for the discrete velocity is $\boldsymbol{V}_h = (V_h)^d$, with:

$$V_h = \left\{ v_h \in \mathrm{L}^2(\Omega) : \forall K \in \mathcal{M},\ v_h|_K \text{ affine};\ \forall \sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L,\ \mathrm{F}_\sigma(v|_K) = \mathrm{F}_\sigma(v|_L);\ \forall \sigma \in \mathcal{E}_{\text{ext}},\ \mathrm{F}_\sigma(v) = 0 \right\}.$$

We denote by ϕ_σ the shape function associated to σ, *i.e.* the unique function of V_h satisfying for any $\sigma' \in \mathcal{E}$:

$$\mathrm{F}_{\sigma'}(\phi_\sigma) = 1 \text{ if } \sigma' = \sigma, \quad \mathrm{F}_{\sigma'}(\phi_\sigma) = 0 \text{ otherwise.}$$

The pressure is approximated by the space $\mathrm{H}_{\mathcal{M}}$ of piecewise constant functions over $\mathcal{M}$:

$$\mathrm{H}_{\mathcal{M}} = \left\{ q_h \in \mathrm{L}^2(\Omega) :\ q_h|_K \text{ constant}, \forall K \in \mathcal{M} \right\}$$

Since only the continuity of the integral over each face of the mesh is imposed, the functions of V_h are discontinuous through each edge; the discretization is thus nonconforming in $\mathrm{H}^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $v \in V_h$, $\partial_{h,i}\, v$ as the function of $\mathrm{L}^2(\Omega)$ which is equal to the (piecewise constant) derivative of v with respect to the i^{th} space variable almost everywhere. This notation allows to define the discrete gradient, denoted by $\boldsymbol{\nabla}_h$, for both scalar and vector valued discrete functions and the discrete divergence of vector valued discrete functions, denoted by $\boldsymbol{\nabla}\cdot_h$.

The Crouzeix-Raviart pair of approximation spaces for the velocity and the pressure is *inf-sup* stable, in the usual sense for piecewise H^1 discrete velocities, *i.e.* there exists $c_{\mathrm{i}} > 0$ independent of the mesh such that:

$$\forall q \in \mathrm{H}_{\mathcal{M}}, \qquad \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{\displaystyle\sum_{K \in \mathcal{M}} \int_K q \ \boldsymbol{\nabla}\cdot\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}}{|\boldsymbol{v}|_{1,b}} = \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{\displaystyle\int_\Omega q \ \boldsymbol{\nabla}\cdot_h\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}}{|\boldsymbol{v}|_{1,b}} \geq c_{\mathrm{i}} \ \|q - q_{\mathrm{m}}\|_{\mathrm{L}^2(\Omega)},$$

where q_{m} is the mean value of q over Ω and $|\cdot|_{1,b}$ stands for the broken Sobolev H^1 semi-norm, which is defined for any function $v \in V_h$ or $v \in \boldsymbol{V}_h$ by:

$$|v|_{1,b}^2 = \sum_{K \in \mathcal{M}} \int_K |\boldsymbol{\nabla} v|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{\nabla}_h v|^2 \,\mathrm{d}\boldsymbol{x}.$$

This broken Sobolev semi-norm is known to control the L^2 norm by an extended Poincaré inequality [19, proposition 4.13]:

Lemma 3.2.2. *For any function $v \in V_h$, there exists a real number $c_{\mathrm{p}}(\Omega) > 0$ only depending on the domain Ω such that:*

$$\|v\|_{\mathrm{L}^2(\Omega)} \leq c_p(\Omega) \ |v|_{1,b}$$

We denote by r_h the following interpolation operator:

$$r_h : \quad \left| \begin{array}{lll} \mathrm{H}_0^1(\Omega) & \longrightarrow & V_h \\ v & \mapsto & r_h v = \displaystyle\sum_{\sigma \in \mathcal{E}} F_\sigma(v)\, \phi_\sigma = \sum_{\sigma \in \mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \,\mathrm{d}\gamma \right) \phi_\sigma. \end{array} \right. \tag{3.6}$$

This operator naturally extends to vector-valued functions (*i.e.* to perform the interpolation from $\mathrm{H}_0^1(\Omega)^d$ to $\boldsymbol{V}_h$), and we keep the same notation r_h for both the scalar and vector case. The properties of r_h are gathered in the following lemma. They are proven in [8].

Lemma 3.2.3. *Let $\theta_0 > 0$ and let $\mathcal{M}$ be a triangulation of the computational domain Ω such that $\theta_{\mathcal{M}} \geq \theta_0$, where $\theta_{\mathcal{M}}$ is defined by* (3.4)*. The interpolation operator r_h enjoys the following properties:*

1. *Preservation of the divergence:*

$$\forall \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d,\ \forall q \in \mathrm{H}_{\mathcal{M}}, \qquad \int_\Omega q \ \boldsymbol{\nabla}\cdot_h(r_h \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega q \ \boldsymbol{\nabla}\cdot\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}. \tag{3.7}$$

2. *Stability:*

$$\forall v \in \mathrm{H}_0^1(\Omega), \qquad |r_h v|_{1,b} \leq c_1(\theta_0) \ |v|_{\mathrm{H}^1(\Omega)}. \tag{3.8}$$

3. *Approximation properties:*

$$\begin{aligned} &\forall v \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega),\ \forall K \in \mathcal{M}, \\ &\qquad \|v - r_h v\|_{\mathrm{L}^2(K)} + h_K \|\boldsymbol{\nabla}(v - r_h v)\|_{\mathrm{L}^2(K)} \leq c_2(\theta_0)\, h_K^2 \ \|v\|_{\mathrm{H}^2(K)}. \end{aligned} \tag{3.9}$$

In the above inequalities, the notation $c_i(\theta_0)$ means that the real number c_i only depends on θ_0, and, in particular, not on the parameter $h_{\mathcal{M}}$ characterizing the size of the cells.

The following compactness result of a family of discrete functions of V_h in $\mathrm{L}^2(\Omega)$ has been proven in [12].

Theorem 3.2.4 (Compactness of a sequence of discrete functions in $\mathrm{L}^2(\Omega)$)**.**
Let $\left(\mathcal{M}^{(m)}\right)_{m \in \mathbb{N}}$ be a regular sequence of discretizations of Ω, in the sense of Definition 3.2.1. Let $\left(v^{(m)}\right)_{m \in \mathbb{N}}$ be a sequence of functions such that $\forall n \in \mathbb{N}$, $v^{(m)} \in V_h^{(m)}$. If there exists a constant real number $C > 0$ such that:

$$\forall\, m \in \mathbb{N}, \qquad \|v\|_{1,b} \leq C,$$

then the sequence $\left(v^{(m)}\right)_{m \in \mathbb{N}}$ converges (up to the extraction of a subsequence) in $\mathrm{L}^2(\Omega)$ as $m \to \infty$ to a limit $v \in \mathrm{H}_0^1(\Omega)$.

The following technical lemma can be found in [12, Lemma 2.4]

Lemma 3.2.5. *Let $\theta_0 > 0$ and let $\mathcal{M}$ be a triangulation of the computational domain Ω such that $\theta_{\mathcal{M}} \geq \theta_0$, where $\theta_{\mathcal{M}}$ is defined by (3.4); let $(a_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ be a family of real numbers such that $\forall\sigma \in \mathcal{E}_{\rm int},\ |a_\sigma| \leq 1$ and let v be a function of the Crouzeix-Raviart space V_h associated to $\mathcal{M}$. Then the following bound holds:*

$$\sum_{\sigma\in\mathcal{E}_{\rm int}} \left| \int_\sigma a_\sigma\ [v]\,\varphi\,{\rm d}\gamma \right| \leq c(\theta_0)\ h\ |v|_{1,b}\ |\varphi|_{{\rm H}^1(\Omega)},\ \forall\varphi \in {\rm H}^1_0(\Omega),$$

where the real number $c(\theta_0)$ only depends on θ_0 and on the domain Ω.

We are now in position to prove the following weak convergence result.

Lemma 3.2.6 (Weak convergence in $\mathrm{L}^2(\Omega)^d$ of the discrete gradients)**.**
Let $\left(\mathcal{M}^{(m)}\right)_{m\in\mathbb{N}}$ be a regular sequence of discretizations of Ω, in the sense of Definition 3.2.1. Let $\left(v^{(m)}\right)_{m\in\mathbb{N}}$ be a sequence of discrete functions (i.e. such that for any $m \in \mathbb{N}$, $v^{(m)}$ belongs to $V_h^{(m)}$) such that:

$$\forall\, m \in \mathbb{N}, \qquad \|v\|_{1,b} \leq C,$$

where C is a positive real number. Let us suppose that the sequence $\left(v^{(m)}\right)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^2(\Omega)$ to $v \in \mathrm{H}^1_0(\Omega)$. Then the sequence $\left(\boldsymbol{\nabla}_h v^{(m)}\right)_{m\in\mathbb{N}}$ of piecewise constant discrete gradients converges weakly in $\mathrm{L}^2(\Omega)^d$ to $\boldsymbol{\nabla} v$.

Proof. Let $\varphi \in \mathrm{C}^\infty_c(\Omega)$. Let $m \in \mathbb{N}$ and $1 \leq i \leq d$. Then, integrating by parts, we get:

$$\int_\Omega \varphi\ \partial_{i,h} v^{(m)}\,{\rm d}\boldsymbol{x} = \sum_{K\in\mathcal{M}^{(m)}} \int_K \varphi\ \partial_i v^{(m)}\,{\rm d}\boldsymbol{x} = -\sum_{K\in\mathcal{M}^{(m)}} \int_K v^{(m)}\ \partial_i\varphi\,{\rm d}\boldsymbol{x} + \sum_{K\in\mathcal{M}^{(m)}} \int_{\partial K} (\boldsymbol{n}_K)_i\ v^{(m)}\ \varphi\,{\rm d}\gamma,$$

where ∂K stands for the boundary of K and $(\boldsymbol{n}_K)_i$ for the i^{th} component of the outward normal to K. Reordering the sums, the second term reads:

$$\sum_{K\in\mathcal{M}^m} \int_{\partial K} (\boldsymbol{n}_K)_i\ v^{(m)}\ \varphi\,{\rm d}\gamma = \sum_{\sigma\in\mathcal{E}^{(m)}_{\rm int}} \int_\sigma (\boldsymbol{n}_{K,\sigma})_i\ [v^{(m)}]\,\varphi\,{\rm d}\gamma.$$

By Lemma 3.2.5, this term tends to zero when m tends to infinity. We thus get:

$$\lim_{m\to\infty} \int_\Omega \varphi\ \partial_{i,h} v^{(m)}\,{\rm d}\boldsymbol{x} = -\lim_{m\to\infty} \int_\Omega v^{(m)}\ \partial_i\varphi\,{\rm d}\boldsymbol{x} = -\int_\Omega v\ \partial_i\varphi\,{\rm d}\boldsymbol{x} = \int_\Omega \varphi\ \partial_i v\,{\rm d}\boldsymbol{x}.$$

The conclusion follows by density, since the sequence $\left(\partial_{i,h} v^{(m)}\right)_{m\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$. □

3.2.3 Finite volumes: discrete space and functional analysis

For a finite $\alpha \geq 1$, we define a discrete $\mathrm{W}^{1,\alpha}_0$–norm on $\mathrm{H}_{\mathcal{M}}$, the space of piecewise constant functions over any element $K \in \mathcal{M}$, by:

$$\forall v \in \mathrm{H}_{\mathcal{M}}, \quad \|v\|^\alpha_{1,\alpha,\mathcal{M}} = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} |\sigma|\, d_\sigma \left|\frac{v_K - v_L}{d_\sigma}\right|^\alpha + \sum_{\sigma\in\mathcal{E}_{\rm ext},\ \sigma\in\mathcal{E}(K)} |\sigma|\, d_\sigma \left|\frac{v_K}{d_\sigma}\right|^\alpha.$$

As a consequence of the discrete Hölder inequality, the following bound holds for any $\alpha,\ \beta \in [1, +\infty)$ such that $\alpha \leq \beta$:

$$\forall v \in \mathrm{H}_{\mathcal{M}}, \qquad \|v\|_{1,\alpha,\mathcal{M}} \leq (d\,|\Omega|)^{1/\alpha-1/\beta}\ \|v\|_{1,\beta,\mathcal{M}}. \tag{3.10}$$

The following discrete Sobolev inequalities are proven in [9, Lemma 9.5, p.790] and [7, 10].

Lemma 3.2.7 (Discrete Sobolev inequality). *Let $\theta_0 > 0$ and let $\mathcal{M}$ be a triangulation of the computational domain Ω such that $\theta_{\mathcal{M}} \geq \theta_0$, where $\theta_{\mathcal{M}}$ is defined by (3.4). For any $\alpha \in [1, d)$, there exists a real number $C(\Omega, \theta_0, \alpha) > 0$ such that:*

$$\forall v \in \mathrm{H}_{\mathcal{M}}, \qquad \|v\|_{\mathrm{L}^{\alpha^\star}(\Omega)} \leq C(\Omega, \theta_0, \alpha)\ \|v\|_{1,\alpha,\mathcal{M}} \qquad \text{with} \quad \alpha^\star = \frac{d\,\alpha}{d-\alpha}.$$

For $\alpha \geq d$ and any $\beta \in [1, +\infty)$, there exists a real number $C(\Omega, \theta_0, \beta) > 0$ such that:

$$\forall v \in \mathrm{H}_{\mathcal{M}}, \qquad \|v\|_{\mathrm{L}^{\beta}(\Omega)} \leq C(\Omega, \theta_0, \beta)\ \|v\|_{1,\alpha,\mathcal{M}}.$$

In addition, the following bound is proven in [10, Lemma 5.4]

Lemma 3.2.8 (Space translates estimates). *Let $v \in \mathrm{H}_{\mathcal{M}}$, and let $\bar{v}$ be its extension by 0 to $\mathbb{R}^d$. Then:*

$$\|\bar{v}(\cdot + \boldsymbol{y}) - \bar{v}(\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)} \leq \sqrt{d}\ |\boldsymbol{y}|\ \|v\|_{1,1,\mathcal{M}}, \quad \forall \boldsymbol{y} \in \mathbb{R}^d.$$

The following result is a consequence of the Kolmogorov's theorem and of this inequality.

Theorem 3.2.9. *Let $\left(\mathcal{M}^{(m)}\right)_{m\in\mathbb{N}}$ be a regular sequence of discretizations of Ω, in the sense of Definition 3.2.1. Let $\alpha \in [1, +\infty)$, and let $\left(v^{(m)}\right)_{m\in\mathbb{N}}$ be a sequence of discrete functions (i.e. such that, $\forall m \in \mathbb{N}$, $v^{(m)} \in \mathrm{H}_{\mathcal{M}}^{(m)}$, where $\mathrm{H}_{\mathcal{M}}^{(m)}$ is the discrete space associated to $\mathcal{M}^{(m)}$) satisfying:*

$$\forall m \in \mathbb{N}, \qquad \|v^{(m)}\|_{1,\alpha,\mathcal{M}} \leq C,$$

where C is a given positive real number. Then, possibly up to the extraction of a subsequence, the sequence $\left(v^{(m)}\right)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^\beta(\Omega)$ to a function $v \in \mathrm{W}_0^{1,\alpha}(\Omega)$, for any $\beta \in [1, \alpha^\star)$, where $\alpha^\star = d\,\alpha/(d-\alpha)$ if $\alpha < d$ and $\alpha^\star = +\infty$ otherwise.

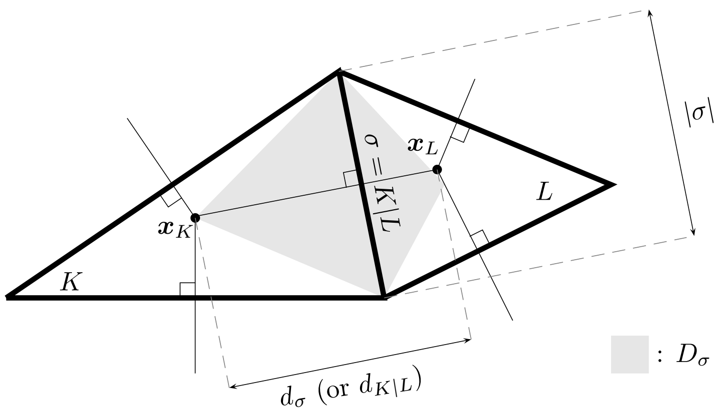

Figure 3.1: Geometrical quantities associated to the mesh

Let us now define a discrete finite volume gradient. To this purpose, for $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex $\boldsymbol{x}_K$. Then for the internal face $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we denote by D_σ the so-called "diamond cell" associated to σ and defined by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$. For an external face $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$, we set $D_\sigma = D_{K,\sigma}$. Finally, for any $v \in \mathrm{H}_{\mathcal{M}}$, we define the discrete gradient $\boldsymbol{\nabla}_{\mathcal{M}} v$ as:

$$\text{for any } \sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L, \qquad \boldsymbol{\nabla}_{\mathcal{M}} v(\boldsymbol{x}) = d\ \frac{v_L - v_K}{d_\sigma}\ \boldsymbol{n}_{K,\sigma}, \quad \forall \boldsymbol{x} \in D_\sigma,$$

$$\text{for any } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K), \qquad \boldsymbol{\nabla}_{\mathcal{M}} v(\boldsymbol{x}) = d\ \frac{0 - v_K}{d_\sigma}\ \boldsymbol{n}_{K,\sigma}, \quad \forall \boldsymbol{x} \in D_\sigma.$$

Lemma 3.2.10 (Weak convergence of the finite volume gradient)**.** *Let $\big(\mathcal{M}^{(m)}\big)_{m\in\mathbb{N}}$ be a regular sequence of discretizations of Ω, in the sense of Definition 3.2.1. Let $\big(v^{(m)}\big)_{m\in\mathbb{N}}$ be a sequence of discrete functions (i.e. such that, $\forall m \in \mathbb{N}$, $v^{(m)} \in \mathrm{H}_{\mathcal{M}}^{(m)}$, where $\mathrm{H}_{\mathcal{M}}^{(m)}$ is the discrete space associated to $\mathcal{M}^{(m)}$). Let us assume that there exists $\alpha \in [1,+\infty)$ and $C > 0$ such that $\|v^{(m)}\|_{1,\alpha,\mathcal{M}} \leq C$, and that $\big(v^{(m)}\big)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^1(\Omega)$ to $v \in \mathrm{W}_0^{1,\alpha}(\Omega)$. Then $\big(\boldsymbol{\nabla}_{\mathcal{M}} v^{(m)}\big)_{m\in\mathbb{N}}$ converges to $\boldsymbol{\nabla} v$ weakly in $\mathrm{L}^\alpha(\Omega)^d$.*

Proof. First, we remark that, by definition, the sequence $\big(\boldsymbol{\nabla}_{\mathcal{M}} v^{(m)}\big)_{m\in\mathbb{N}}$ is bounded in $\mathrm{L}^\alpha(\Omega)^d$. Then, let $\boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)^d$. Since the size $h^{(m)}$ of the sequence of meshes tends to zero, for m large enough, the intersection of the support of $\boldsymbol{\varphi}$ and the diamond cells associated to the external faces of the meshes is reduced to the empty set. For such an m, we have:

$$\int_\Omega \boldsymbol{\nabla}_{\mathcal{M}} v^{(m)} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \sum_{\mathcal{E}\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L} d\, \frac{v_L^{(m)} - v_K^{(m)}}{d_\sigma} \int_{D_\sigma} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \cdot \boldsymbol{n}_{K,\sigma} = T_1^{(m)} + T_2^{(m)},$$

with:

$$T_1^{(m)} = \sum_{\mathcal{E}\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L} (v_L^{(m)} - v_K^{(m)}) \int_\sigma \boldsymbol{\varphi}\cdot\boldsymbol{n} \,\mathrm{d}\gamma,$$

$$T_2^{(m)} = \sum_{\mathcal{E}\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L} |\sigma|\ (v_L^{(m)} - v_K^{(m)}) \ \Big[\frac{1}{|D_\sigma|}\int_{D_\sigma} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} - \frac{1}{|\sigma|}\int_\sigma \boldsymbol{\varphi} \,\mathrm{d}\gamma\Big] \cdot \boldsymbol{n}_{K,\sigma}.$$

Reordering the equations, we get for T_1:

$$T_1^{(m)} = -\sum_{K\in\mathcal{M}} v_K^{(m)} \sum_{\sigma=K|L} \int_\sigma \boldsymbol{\varphi}\cdot\boldsymbol{n}_{K,\sigma} \,\mathrm{d}\gamma = -\sum_{K\in\mathcal{M}} \int_K v^{(m)} \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = -\int_\Omega v^{(m)} \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}.$$

By the weak convergence of $\big(v^{(m)}\big)_{m\in\mathbb{N}}$ in $\mathrm{L}^1(\Omega)$, we thus get:

$$\lim_{n\to+\infty} T_1^{(m)} = -\int_\Omega v\ \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}.$$

On the other hand, thanks to the regularity of $\boldsymbol{\varphi}$, we get:

$$|T_2^{(m)}| \leq c_{\boldsymbol{\varphi}} \|v^{(m)}\|_{1,1,\mathcal{M}}\ h^{(m)},$$

and so, thanks to the control on $\|v^{(m)}\|_{1,1,\mathcal{M}}$ obtained from (3.10), the term tends to zero when m tends to $+\infty$. The results follows by density of $\mathrm{C}_c^\infty(\Omega)^d$ in $\mathrm{L}^\alpha(\Omega)^d$. □

3.3 The numerical scheme

The considered numerical scheme for the discretization of Problem (3.1) combines a standard finite element discretization of the momentum balance equation and an upwind finite volume scheme for the equation satisfied by the turbulent scale k:

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \qquad \int_\Omega \lambda(k)\ \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \boldsymbol{\nabla}_h\cdot\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \tag{3.11a}$$

$$\forall q \in \mathrm{H}_{\mathcal{M}}, \qquad \int_\Omega q\ \boldsymbol{\nabla}\cdot_h \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = 0, \tag{3.11b}$$

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\ \lambda(k)_\sigma\ \big(k_K - k_L\big) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_{K,\sigma}}\ \lambda(k)_\sigma\ \big(k_K\big) + \sum_{\sigma=K|L} \big(\mathrm{v}_{\sigma,K}^+\, k_K - \mathrm{v}_{\sigma,K}^-\, k_L\big) = |K|\ \Big[\lambda(k)\ |\boldsymbol{\nabla}\boldsymbol{u}|^2\Big]_K, \tag{3.11c}$$

where $\mathrm{v}_{K,\sigma}$ approximates the flux of $\boldsymbol{u}$ across the internal face $\sigma = K|L$ outward the element K, and is defined by:

$$\mathrm{v}_{K,\sigma} = \int_{\sigma=K|L} \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\gamma(\boldsymbol{x}),$$

$\mathrm{v}^+_{\sigma,K} = \max(\mathrm{v}_{K,\sigma}, 0)$ and $\mathrm{v}^-_{\sigma,K} = -\min(\mathrm{v}_{K,\sigma}, 0)$ (so $\mathrm{v}_{K,\sigma} = \mathrm{v}^+_{\sigma,K} - \mathrm{v}^-_{\sigma,K}$), the discretization of the source term in (3.11c) reads:

$$\Big[\lambda\,|\boldsymbol{\nabla}_h \boldsymbol{u}|^2\Big]_K = \frac{\lambda(k_K)}{|K|} \int_K |\boldsymbol{\nabla}_h \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x}.$$

and, for $\sigma \in \mathcal{E}$, λ_σ stands for a reasonable approximation of the viscosity on σ, supposed to satisfy:

$$\begin{aligned} &\forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L, && \min\big[\lambda(k_K), \lambda(k_L)\big] \le \lambda(k)_\sigma \le \max\big[\lambda(k_K), \lambda(k_L)\big], \\ &\forall \sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}(K) && \lambda(k)_\sigma = \lambda(k_K). \end{aligned} \tag{3.12}$$

For instance, on $\sigma = K|L$, λ_σ may be defined as the arithmetic or harmonic mean value of $\lambda(k_K)$ and $\lambda(k_L)$.

Finally, in order to obtain a unique solution, we impose:

$$\int_\Omega p \,\mathrm{d}\boldsymbol{x} = 0. \tag{3.13}$$

3.4 *A priori* estimates and existence of a solution to the scheme

We begin this section with recalling two technical lemmas. Both are obtained by algebraic manipulation of discrete summations.

Lemma 3.4.1. *Let us assume that:*

$$\forall K \in \mathcal{M}, \qquad \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma} = 0.$$

Let φ be a function defined over $\mathbb{R}^+$, and let $(k_K)_{K\in\mathcal{M}}$ be a positive function of $\mathrm{H}_\mathcal{M}$. Then the following identity holds:

$$\sum_{K\in\mathcal{M}} \varphi(k_K) \Big[\sum_{\sigma=K|L} \big(\mathrm{v}^+_{\sigma,K}\, k_K - \mathrm{v}^-_{\sigma,K}\, k_L\big) \Big] = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L} |\mathrm{v}_{K,\sigma}|\ (k_K - k_L)\ \big[\varphi(k_K) - \varphi(k_L)\big].$$

As a consequence, if the function φ is non-decreasing, this quantity is non-negative.

Let ψ be the real function defined over $\mathbb{R}^+$ by:

$$\forall s \in \mathbb{R}^+, \qquad \psi(s) = \int_0^s \frac{1}{1+t^\beta} \,\mathrm{d}t \tag{3.14}$$

where $\beta \in (1,2)$. The function ψ is non-negative, increasing and bounded over $\mathbb{R}^+$ by a quantity depending on β (and which tends to $+\infty$ when β tends to 1). In addition, the following estimate is stated and proven in [13, Lemma 5.2] (and was also used in [11], with a less detailed exposition).

Lemma 3.4.2. *Let $\mathcal{M}$ be a discretization of the domain Ω and let $\theta_0 > 0$ be a real number such that $\theta_\mathcal{M} \ge \theta_0$, with $\theta_\mathcal{M}$ defined in (3.4). Let k be a positive function of $\mathrm{H}_\mathcal{M}$ and ψ be the real function defined over $\mathbb{R}^+$ by (3.14). Let $T_d(k)$ be given by:*

$$T_d(k) = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (k_K - k_L) \big[\psi(k_K) - \psi(k_L)\big] + \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{ext}},\\ \sigma\in\mathcal{E}(K)}} \frac{|\sigma|}{d_\sigma} k_K\, \psi(k_K).$$

Then the following bound holds for $1 \le \alpha < 2$:

$$\|k\|^\alpha_{1,\alpha,\mathcal{M}} \le \big[T_d(k)\big]^{\alpha/2} \Big[C_1 + C_2\, \|k\|^{\beta\alpha/2}_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}\Big],$$

where C_1 and C_2 only depend on α and θ_0.

The solution of (3.11) satisfies the following *a priori* estimates.

Theorem 3.4.3. *Let $\mathcal{M}$ be a discretization of the domain Ω and let $\theta_0 > 0$ be a real number such that $\theta_{\mathcal{M}} \geq \theta_0$, with $\theta_{\mathcal{M}}$ defined in* (3.4). *Then a solution $(\boldsymbol{u}, p, k) \in \boldsymbol{V}_h \times \mathrm{H}_{\mathcal{M}} \times \mathrm{H}_{\mathcal{M}}$ to the discrete Problem* (3.11) *satisfies the following estimate:*

$$\|\boldsymbol{u}\|_{1,b} + \|k\|_{1,\alpha,\mathcal{M}} + \|p\|_{\mathrm{L}^2(\Omega)} \leq C,$$

where α is any real number satisfying $1 \leq \alpha < d/(d-1)$ and C only depends on $\boldsymbol{f}$, Ω, μ, θ_0 and α.

Proof. First of all, we remark that the right-hand side of (3.11c) is always (*i.e.* whatever the velocity $\boldsymbol{u}$ may be) non-negative and, if we exclude the trivial case $\boldsymbol{f} = 0$ (so $\boldsymbol{u} \neq 0$), cannot vanish uniformly; as a consequence, from the properties of the upwind finite volume scheme, we get:

$$k_K > 0, \quad \forall K \in \mathcal{M},$$

which ensures, in particular, that $\lambda(k)$ is correctly defined. Taking the solution $\boldsymbol{u}$ as test function in (3.11a) yields:

$$\int_\Omega \lambda(k)\ |\boldsymbol{\nabla}_h \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}.$$

Since, from its definition (3.2), $\lambda(k) \geq \mu$, by the Poincaré inequality 3.2.2, there exists a constant $C_1 > 0$ depending on Ω and $\boldsymbol{f}$ such that:

$$\mu\ \|\boldsymbol{u}\|_{1,b}^2 \leq \int_\Omega \lambda(k)\ |\boldsymbol{\nabla}_h \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \leq C_1. \tag{3.15}$$

The right-hand side of Equation (3.11c) thus satisfies an $\mathrm{L}^1(\Omega)$-estimate. We thus obtain a bound for k using the arguments used in [11] for the Laplace equation with L^1 data. Let $\beta \in (1,2)$ and ψ be the function defined by (3.14); testing Equation (3.11c) against function $\psi(k)$, we get $T_1 + T_2 = T_3$ with:

$$\begin{aligned}
T_1 = &\sum_{K\in\mathcal{M}} \psi(k_K)\ \Big[\sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\ \lambda(k)_\sigma\ \big(k_K - k_L\big) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \frac{|\sigma|}{d_{K,\sigma}}\ \lambda(k)_\sigma\ k_K\Big],\\
T_2 = &\sum_{K\in\mathcal{M}} \psi(k_K)\ \Big[\sum_{\sigma=K|L} \big(\mathrm{v}^+_{\sigma,K}\, k_K - \mathrm{v}^-_{\sigma,K}\, k_L\big)\Big],\\
T_3 = &\sum_{K\in\mathcal{M}} |K|\ \psi(k_K)\ \Big[\lambda(k)\ |\boldsymbol{\nabla}\boldsymbol{u}|^2\Big]_K.
\end{aligned}$$

Since ψ is an increasing function, Lemma 3.4.1 yields $T_2 \geq 0$. The term T_3 reads:

$$T_3 = \int_\Omega \psi(k)\ \lambda(k)\ |\boldsymbol{\nabla}_h \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x},$$

thus, combining the fact that ψ is bounded and Inequality (3.15), we get $T_3 \leq C_2$, where C_2 depends on β, $\boldsymbol{f}$ and Ω. Reordering the summation, we get:

$$T_1 = \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \lambda_\sigma\ \frac{|\sigma|}{d_\sigma}\,(k_K - k_L)\,\big[\psi(k_K) - \psi(k_L)\big] + \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext},\\ \sigma\in\mathcal{E}(K)}} \lambda_\sigma\ \frac{|\sigma|}{d_\sigma}\, k_K\, \psi(k_K).$$

Since ψ is an increasing function, T_1 is a sum of non-negative terms and, from the definition (3.12) of λ_σ, we get $T_1 \geq \mu\ T_d(k)$, where $T_d(k)$ is the quantity defined in Lemma 3.4.2. Gathering these estimates, we obtain $\mu\ T_d(k) \leq C_3$, and thus, by Lemma 3.4.2:

$$\|k\|_{1,\alpha,\mathcal{M}}^\alpha \leq C_4 + C_5\ \|k\|_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}^{\beta\alpha/2},$$

where $\alpha \in [1,2)$ and C_3, C_4 and C_5 only depend on $\boldsymbol{f}$, μ, Ω, α and θ_0. If we now choose α in such a way that the $\|\cdot\|_{1,\alpha,\mathcal{M}}$ controls the $\|\cdot\|_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}$ norm, we obtain an inequality of the form:

$$\|k\|_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}^\alpha \leq C_4 + C_5\ \|k\|_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}^{\beta\alpha/2},$$

which, since $\beta < 2$ and thus $\alpha > \beta\alpha/2$, yields a control on $\|k\|_{\mathrm{L}^{\beta\alpha/(2-\alpha)}(\Omega)}$ and then, returning to the first inequality, on $\|k\|_{1,\alpha,\mathcal{M}}$. From the discrete Sobolev inequality of Lemma 3.2.7, α must satisfy:

$$\frac{d\alpha}{d-\alpha} \geq \frac{\beta\alpha}{2-\alpha},$$

which may be set under the form $\alpha \leq \alpha_0(\beta)$, with $\alpha_0(\beta) = d\,(2-\beta)/(d-\beta)$. We observe that $\alpha_0(\beta)$ decrases with β and tends to $d/(d-1)$ when β tends to 1, which yields the desired estimate for k.

Finally, from the *inf-sup* stability of the Crouzeix-Raviart element, since, by (3.13), the pressure p is a zero mean-valued function, there exists C_6 only depending on Ω such that:

$$\begin{aligned} \|p\|_{\mathrm{L}^2(\Omega)} &\leq C_6 \sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{1}{\|\boldsymbol{v}\|_{1,b}} \int_\Omega p\ \boldsymbol{\nabla}\cdot_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \\ &= C_6 \sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{1}{\|\boldsymbol{v}\|_{1,b}} \Big[\int_\Omega \lambda(k)\ \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\Big], \end{aligned}$$

which, invoking the fact that $\lambda(k) \leq \lambda_\infty$, the estimate for $\boldsymbol{u}$, the Cauchy-Schwarz and the Poincaré inequality, concludes the proof. ☐

Theorem 3.4.4. *The discrete problem* (3.11) *admits at least one solution.*

Proof. Let us consider the iteration consisting in solving the system (3.16a)-(3.16b) of equations below for a pair $(\boldsymbol{u}^{\ell+1}, p^{\ell+1}) \in \boldsymbol{V}_h \times \bar{\mathrm{H}}_{\mathcal{M}}$, where $\bar{\mathrm{H}}_{\mathcal{M}}$ stands for the space of zero mean-valued functions of $\mathrm{H}_{\mathcal{M}}$, then solving Equation (3.16c) for $k^{\ell+1} \in \mathrm{H}_{\mathcal{M}}$:

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h,\qquad \int_\Omega \lambda(k^\ell)\ \boldsymbol{\nabla}_h \boldsymbol{u}^{\ell+1} : \boldsymbol{\nabla}_h \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p^{\ell+1}\ \boldsymbol{\nabla}_h\cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \tag{3.16a}$$

$$\forall q \in \mathrm{H}_{\mathcal{M}},\qquad \int_\Omega q\ \boldsymbol{\nabla}\cdot_h \boldsymbol{u}^{\ell+1}\,\mathrm{d}\boldsymbol{x} = 0, \tag{3.16b}$$

$$\begin{aligned} \forall K \in \mathcal{M},\quad \sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\ \lambda(k^\ell)_\sigma\ \big(k_K^{\ell+1} - k_L^{\ell+1}\big) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \frac{|\sigma|}{d_{K,\sigma}}\ \lambda(k^\ell)_\sigma\ \big(k_K^{\ell+1}\big) \\ + \sum_{\sigma=K|L} \big(\mathrm{v}_{\sigma,K}^+\, k_K^{\ell+1} - \mathrm{v}_{\sigma,K}^-\, k_L^{\ell+1}\big) = |K|\ \Big[\lambda(k^\ell)\ |\boldsymbol{\nabla}\boldsymbol{u}^{\ell+1}|^2\Big]_K. \end{aligned} \tag{3.16c}$$

This iteration is performed for $\ell \geq 1$ and initialized by $k^0 = 0$. Each subproblem (*i.e.* (3.16a)-(3.16b) on one side and (3.16c) on the other side) is linear, and, at each step, $\boldsymbol{u}^{\ell+1}$, $p^{\ell+1}$ and $k^{\ell+1}$ satisfy, by the same arguments, the estimates of Theorem 3.4.3. The Brouwer theorem thus applies, and the iteration admits a fixed point, which is solution to the scheme. ☐

3.5 Convergence analysis

The aim of this section is to prove the convergence of a sequence of discrete solutions to a (weak) solution to the continuous problem. We begin with proving that such a sequence indeed has a limit, thanks to compactness arguments; next steps, performed in the following lemmas, consist in passing to the limit in the scheme.

Lemma 3.5.1 (Compactness of discrete solutions)**.** *Let* $\big(\mathcal{M}^{(m)}\big)_{m\in\mathbb{N}}$ *be a regular sequence of discretizations of* Ω*, in the sense of Definition 3.2.1. Let* $(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)})_{m\in\mathbb{N}}$ *be the corresponding sequence of approximate solutions (i.e. , for a given* m*,* $(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)})$ *is the solution to Problem* (3.11) *with the mesh* $\mathcal{M}^{(m)}$*). Then, up to the extraction of a subsequence:*

1. *the sequence* $\left(\boldsymbol{u}^{(m)}\right)_{m\in\mathbb{N}}$ *converges in* $\mathrm{L}^2(\Omega)^d$*, to a limit* $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$*, and the sequence of discrete gradients* $\left(\boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}\right)_{m\in\mathbb{N}}$ *weakly converges in* $\mathrm{L}^2(\Omega)^{d\times d}$ *to* $\boldsymbol{\nabla}\boldsymbol{u}$*;*
2. $\left(p^{(m)}\right)_{m\in\mathbb{N}}$ *weakly converges in* $\mathrm{L}^2(\Omega)$ *to* $p \in \mathrm{L}^2(\Omega)$*;*
3. $\left(k^{(m)}\right)_{m\in\mathbb{N}}$ *converges in* $\mathrm{L}^\beta(\Omega)$ *to* $k \in \mathrm{W}^{1,\alpha}_0(\Omega)$*, for any* $\alpha \in [1, d/(d-1))$ *and* $\beta \in [1, d\,/(d-2))$ *and the sequence of finite volume gradients* $\left(\boldsymbol{\nabla}_{\mathcal{M}} k^{(m)}\right)_{m\in\mathbb{N}}$ *weakly converges to* $\boldsymbol{\nabla} k$ *in* $\mathrm{L}^\alpha(\Omega)^d$*, for any* $\alpha \in [1, d/(d-1))$*.*

Proof. These convergence properties are straightforward consequences of estimates of Theorem 3.4.3, compactness Theorems 3.2.4 and 3.2.9 and Lemmas 3.2.6 and 3.2.10. □

Lemma 3.5.2 (Strong convergence of $\lambda(k^{(m)})$**).** *Under the assumptions of Lemma 3.5.1, the sequence* $\lambda(k^{(m)})_{m\in\mathbb{N}}$ *converges in* $\mathrm{L}^\beta(\Omega)$ *to* $\lambda(k)$*, for* $\beta \in [1, +\infty)$*.*

Proof. With the specific form of λ, we have, for any $\xi \in (0,1)$:

$$\forall s_1,\ s_2 \in (0,+\infty),\quad |\lambda(s_1) - \lambda(s_2)| = |\lambda(s_1) - \lambda(s_2)|^{1-\xi}\ |\lambda(s_1) - \lambda(s_2)|^{\xi} \le \lambda_\infty^{1-\xi}\ \left[\frac{\ell^2}{2\mu}\right]^{\xi}\ |s_1 - s_2|^{\xi}.$$

Let $\beta \in [1,+\infty)$ be given. For $m \in \mathbb{N}$, choosing $\xi = 1/\beta$ in the above inequality, we get:

$$\|\lambda(k^{(m)}) - \lambda(k)\|_{\mathrm{L}^\beta(\Omega)} \le \lambda_\infty^{1-1/\beta}\ \left[\frac{\ell^2}{2\mu}\right]^{1/\beta}\ \|k^{(m)} - k\|^{1/\beta}_{\mathrm{L}^1(\Omega)}.$$

and the convergence follows from the (strong) convergence of $\left(k^{(m)}\right)_{m\in\mathbb{N}}$ in $\mathrm{L}^1(\Omega)$. □

Remark 3.5.3. The same convergence result may be obtained supposing only that the function λ is continuous and bounded. Indeed, since $\left(k^{(m)}\right)_{m\in\mathbb{N}}$ converges to k in $\mathrm{L}^1(\Omega)$, we know that a subsequence, still denoted $\left(k^{(m)}\right)_{m\in\mathbb{N}}$, converges to k *almost everywhere* in Ω. If λ is continuous, the same *almost everywhere* convergence holds for $\lambda(k^{(m)})_{m\in\mathbb{N}}$, now to $\lambda(k)$, and the desired convergence in $\mathrm{L}^\beta(\Omega)$ follows by Lebesgue Dominated Convergence Theorem. Finally, since the limit is unique, the whole sequence converges.

Lemma 3.5.4 (The limit satisfies the Stokes equations)**.** *Under the assumptions of Lemma 3.5.1, the limit* $(\boldsymbol{u}, p, k)$ *satisfies the weak Stokes equations, i.e. the first two equations of the weak continuous problem* (3.3)*.*

Proof. Let $\boldsymbol{\varphi}$ be a function of $\mathrm{C}^\infty_c(\Omega)^d$ and $\boldsymbol{\varphi}^{(m)} = r_h^{(m)}\boldsymbol{\varphi}$ its interpolate in $\boldsymbol{V}_h^{(m)}$, with $r_h^{(m)}$ defined by Relation (3.6). Taking $\boldsymbol{\varphi}_h^{(m)}$ as test function in Equation (3.11a), we get:

$$\underbrace{\int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x}}_{T_1} - \underbrace{\int_\Omega p^{(m)}\ \boldsymbol{\nabla}_h \cdot \boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x}}_{T_2} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x}. \tag{3.17}$$

The diffusion term reads:

$$T_1 = \int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \big(\boldsymbol{\varphi}^{(m)} - \boldsymbol{\varphi}\big)\,\mathrm{d}\boldsymbol{x}.$$

Remarking that $\boldsymbol{\nabla}_h \boldsymbol{\varphi} = \boldsymbol{\nabla}\boldsymbol{\varphi}$, thanks to the strong convergence in $\mathrm{L}^2(\Omega)$ of $\lambda(k^{(m)})$ and the weak convergence of $\boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}$ in $\mathrm{L}^2(\Omega)^{d\times d}$, we get for the first term:

$$\lim_{m\to+\infty} \int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k)\ \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.$$

Since the quantity $\lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}$ is bounded in $\mathrm{L}^2(\Omega)^{d\times d}$ independently of m, we get for the second term, invoking first the Cauchy-Schwarz inequality and then the approximation properties of the interpolation operator:

$$\int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h\big(\boldsymbol{\varphi}^{(m)} - \boldsymbol{\varphi}\big)\,\mathrm{d}\boldsymbol{x} \le \|\lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}\|_{\mathrm{L}^2(\Omega)^{d\times d}}\ \|\boldsymbol{\nabla}_h\big(\boldsymbol{\varphi}^{(m)} - \boldsymbol{\varphi}\big)\|_{\mathrm{L}^2(\Omega)^{d\times d}} \le C_{\boldsymbol{\varphi}}\ h^{(m)},$$

where $C_{\boldsymbol{\varphi}}$ does not depends on m. Consequently, this term tends to zero when m tends to $+\infty$.

Let us now turn to T_2. The weak conservation of the divergence satisfied by the interpolation operator $r_h^{(m)}$ yields:

$$\int_\Omega p^{(m)}\ \boldsymbol{\nabla}_h\cdot\boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x} = \int_\Omega p^{(m)}\ \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x},$$

and thus, since $p^{(m)}$ weakly converges to p in $\mathrm{L}^2(\Omega)$:

$$\lim_{m\to+\infty} \int_\Omega p^{(m)}\ \boldsymbol{\nabla}_h\cdot\boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x} = \int_\Omega p\ \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.$$

Finally, thanks to the interpolation properties of $r_h^{(m)}$:

$$\lim_{m\to+\infty} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}^{(m)}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.$$

Gathering all the terms, we thus obtain that the triplet $(\boldsymbol{u}, p, k)$ satisfies the weak form of the momentum balance equation for any test function in $\mathrm{C}_c^\infty(\Omega)^d$, and this result may be extended to any function of $\mathrm{H}_0^1(\Omega)^d$ by density.

Let us now address the divergence constraint. Let $\varphi \in \mathrm{C}_c^\infty(\Omega)$, and let $\pi^{(m)}\varphi$ be the function of $\mathrm{H}_{\mathcal{M}}^{(m)}$ obtained by taking over each cell of the mesh $\mathcal{M}^{(m)}$ the mean value of φ. Taking $\pi^{(m)}\varphi$ as test function in the discrete divergence constraint, we have:

$$\int_\Omega \pi^{(m)}\varphi\ \boldsymbol{\nabla}_h\cdot\boldsymbol{u}^{(m)}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \varphi\ \boldsymbol{\nabla}_h\cdot\boldsymbol{u}^{(m)}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \big(\pi^{(m)}\varphi - \varphi\big)\ \boldsymbol{\nabla}_h\cdot\boldsymbol{u}^{(m)}\,\mathrm{d}\boldsymbol{x} = 0.$$

Since $\|\pi^{(m)}\varphi - \varphi\|_{\mathrm{L}^2(\Omega)}$ tends to zero as $h^{(m)}$ when m tends to $+\infty$, $\boldsymbol{\nabla}_h\cdot\boldsymbol{u}^{(m)}$ is bounded in $\mathrm{L}^2(\Omega)$ and weakly converges in $\mathrm{L}^2(\Omega)$ to $\boldsymbol{\nabla}\cdot(\boldsymbol{u})$ (since $\boldsymbol{\nabla}_h\boldsymbol{u}^{(m)}$ weakly converges to $\boldsymbol{\nabla}\boldsymbol{u}$), we can pass to the limit in this relation, which yields that $\boldsymbol{u}$ is divergence free. □

Lemma 3.5.5 (Strong convergence of the viscous dissipation)**.** *Under the assumptions of Lemma 3.5.1, the sequence* $(\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ *converges (strongly) in* $\mathrm{L}^2(\Omega)^{d\times d}$ *to* $\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u}$.

Proof. The sequence $(\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)^{d\times d}$ so, up to the extraction of a subsequence, converges to a limit, let us say $\overline{\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u}}$, which thus satisfies:

$$\lim_{m\to+\infty} \int_\Omega \lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u}} : \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{\varphi} \in \mathrm{L}^2(\Omega)^{d\times d}.$$

But, since, as proven in Lemma 3.5.2, the sequence $(\lambda(k^{(m)}))_{m\in\mathbb{N}}$ converges in $\mathrm{L}^\beta(\Omega)$ for any $\beta \in [1,+\infty)$, so does the sequence $(\lambda(k^{(m)})^{1/2})_{m\in\mathbb{N}}$. By the weak convergence of $\big(\boldsymbol{\nabla}_h\boldsymbol{u}^{(m)}\big)_{m\in\mathbb{N}}$ to $\boldsymbol{\nabla}\boldsymbol{u}$, we thus get:

$$\lim_{m\to+\infty} \int_\Omega \lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)^{d\times d},$$

which yields that:

$$\overline{\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u}} = \lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u},$$

Since the weak limit is unique, the whole sequence $(\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ weakly converges.

Let us now take $\boldsymbol{u}^m$ as test function in the first equation of the scheme, to obtain:

$$\int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{(m)}\ \mathrm{d}\boldsymbol{x}.$$

By the convergence of the sequence $\big(\boldsymbol{u}^{(m)}\big)_{m\in\mathbb{N}}$ in $\mathrm{L}^2(\Omega)^d$, we thus get:

$$\lim_{m\to+\infty}\int_\Omega \lambda(k^{(m)})\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)} : \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}.$$

But, by Lemma 3.5.4, we know that the triplet $(\boldsymbol{u},p,k)$ satisfies the weak form of the continuous Stokes problem, which, taking $\boldsymbol{u}$ as test function, yields:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k)\ \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}.$$

The sequence $(\|\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h\boldsymbol{u}^{(m)}\|_{\mathrm{L}^2(\Omega)^{d\times d}})_{m\in\mathbb{N}}$ thus converges to the norm of the weak limit, which proves the strong convergence. □

To make easier the passage to the limit in the turbulent energy balance equation, we begin with defining a piecewise constant viscosity field over the diamond meshes. For $m\in\mathbb{N}$ given, this viscosity field, denoted by $\tilde\lambda^{(m)}$, is the function of $\mathrm{L}^\infty(\Omega)$ given by:

$$\forall\sigma\in\mathcal{E}^{(m)},\ \forall\boldsymbol{x}\in D_\sigma,\qquad \tilde\lambda^{(m)}(\boldsymbol{x}) = \lambda(k^{(m)})_\sigma,$$

where $\lambda(k^{(m)})_\sigma$ is defined by (3.12). This viscosity field statisfies the following convergence result.

Lemma 3.5.6. *Under the assumptions of Lemma 3.5.1, the sequence* $\big(\tilde\lambda^{(m)}\big)_{m\in\mathbb{N}}$ *converges to* $\lambda(k)$ *in* $\mathrm{L}^\beta(\Omega)$*, for any* $\beta\in[1,+\infty)$.

Proof. Let $\beta\in[1,+\infty)$. Since the sequence $(\lambda(k^{(m)}))_{m\in\mathbb{N}}$ converges to $\lambda(k)$ in $\mathrm{L}^\beta(\Omega)$, it is sufficient to show that $\|\lambda(k^{(m)})-\tilde\lambda^{(m)}\|_{\mathrm{L}^\beta(\Omega)}$ tends to zero when m tends to $+\infty$. By definition:

$$\|\lambda(k^{(m)})-\tilde\lambda^{(m)}\|^\beta_{\mathrm{L}^\beta(\Omega)} = \sum_{K\in\mathcal{M}^{(m)}}\ \sum_{\sigma\in\mathcal{E}(K)} |D_{K,\sigma}|\ \big|\lambda(k_K^{(m)}) - \lambda(k^{(m)})_\sigma\big|^\beta,$$

and thus, by the definition of the viscosity at the faces (3.12):

$$\|\lambda(k^{(m)})-\tilde\lambda^{(m)}\|^\beta_{\mathrm{L}^\beta(\Omega)} \le \sum_{\sigma\in\mathcal{E}^{(m)}} |D_\sigma|\ \big|\lambda(k_K^{(m)}) - \lambda(k_L^{(m)})\big|^\beta.$$

Since, for $s_1,\ s_2\ge 0$:

$$|\lambda(s_1)-\lambda(s_2)|^\beta \le \lambda_\infty^{\beta-1}\ |\lambda(s_1)-\lambda(s_2)| \le \frac{\ell^2\lambda_\infty^{\beta-1}}{2\mu}\ |s_1-s_2|,$$

we thus get:

$$\|\lambda(k^{(m)})-\tilde\lambda^{(m)}\|^\beta_{\mathrm{L}^\beta(\Omega)} \le \frac{\ell^2\lambda_\infty^{\beta-1}}{2\mu}\sum_{\sigma\in\mathcal{E}^{(m)}} |D_\sigma|\ \big|k_K^{(m)}-k_L^{(m)}\big| \le \frac{\ell^2\lambda_\infty^{\beta-1}}{2\mu}\ \|k^{(m)}\|_{1,1,\mathcal{M}}\ h^{(m)},$$

which concludes the proof. □

Lemma 3.5.7 ($(\boldsymbol{u},k)$ satisfies the turbulent energy balance)**.** *Under the assumptions of Lemma 3.5.1, the pair* $(\boldsymbol{u},k)$ *satisfies the weak form of the turbulent energy balance equation, i.e. the third relation of the weak continuous problem* (3.3).

Proof. Let $\varphi \in \mathrm{C}_c^\infty(\Omega)$, and let $\pi^{(m)}\varphi$ be the function of $\mathrm{H}_\mathcal{M}^{(m)}$ obtained by taking over each cell K of the mesh $\mathcal{M}^{(m)}$ the value of φ in $\boldsymbol{x}_K$. For m large enough, the intersection between the cells of the mesh $\mathcal{M}^{(m)}$ having an external face and the support of φ is empty. Taking $\pi^{(m)}\varphi$ as test function in the discrete turbulent energy balance (3.11c), we then have:

$$\begin{aligned} &\underbrace{\sum_{K\in\mathcal{M}} \varphi(\boldsymbol{x}_K) \sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\, \lambda(k^{(m)})_\sigma\, \big(k_K^{(m)} - k_L^{(m)}\big)}_{T_d^{(m)}} \\ &\quad + \underbrace{\sum_{K\in\mathcal{M}} \varphi(\boldsymbol{x}_K) \sum_{\sigma=K|L} \big((\mathrm{v}_{\sigma,K}^{+})^{(m)}\, k_K^{(m)} - (\mathrm{v}_{\sigma,K}^{-})^{(m)}\, k_L^{(m)}\big)}_{T_c^{(m)}} = \int_\Omega \lambda(k^{(m)})\, |\boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}|^2\, \pi^{(m)}\varphi \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.18}$$

Reordering the summations, we get:

$$T_d^{(m)} = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} |D_\sigma|\ \lambda(k^{(m)})_\sigma\ d\ \frac{k_K^{(m)} - k_L^{(m)}}{d_\sigma}\ \frac{\varphi(\boldsymbol{x}_K) - \varphi(\boldsymbol{x}_L)}{d_\sigma} = T_{d,1}^{(m)} + T_{d,2}^{(m)},$$

with:

$$\begin{aligned} T_{d,1}^{(m)} &= \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_{D_\sigma} \tilde\lambda^{(m)}\ \boldsymbol{\nabla}_\mathcal{M} k^{(m)} \cdot \boldsymbol{\nabla}\varphi \,\mathrm{d}\boldsymbol{x}, \\ T_{d,2}^{(m)} &= \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} d\ |D_\sigma|\ \lambda(k^{(m)})_\sigma\ \frac{k_K^{(m)} - k_L^{(m)}}{d_\sigma}\ \Big[\frac{\varphi(\boldsymbol{x}_K) - \varphi(\boldsymbol{x}_L)}{d_\sigma} - \frac{1}{|D_\sigma|} \int_{D_\sigma} \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{\nabla}\varphi \,\mathrm{d}\boldsymbol{x}\Big]. \end{aligned}$$

Let us choose $\alpha \in [1, d/(d-1)$ and $\beta \in [1,+\infty)$ in such a way that $1/\alpha + 1/\beta = 1$. By the convergence of $\big(\tilde\lambda^{(m)}\big)_{m\in\mathbb{N}}$ in $\mathrm{L}^\beta(\Omega)$ and the weak convergence of $\big(\boldsymbol{\nabla}_\mathcal{M} k^{(m)}\big)_{m\in\mathbb{N}}$ in $\mathrm{L}^\alpha(\Omega)^d$ we have:

$$\lim_{m\to+\infty} T_{d,1}^{(m)} = \int_\Omega \lambda(k)\ \boldsymbol{\nabla} k \cdot \boldsymbol{\nabla}\varphi \,\mathrm{d}\boldsymbol{x}.$$

On the other hand, thanks to the regularity of φ, we have:

$$|T_{d,2}^{(m)}| \le C_\varphi\ \lambda_\infty\ \|k^{(m)}\|_{1,1,\mathcal{M}}\ h^{(m)},$$

where C_φ does not depend on the mesh, so this term tends to zero when m tends to $+\infty$.

The proof that:

$$\lim_{m\to+\infty} T_c^{(m)} = -\int_\Omega k\ \boldsymbol{u} \cdot \boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x}$$

can be found in [12, Step 3 of proof of Theorem 6.1]; in this latter paper, the authors make the same passage to the limit for the term $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})$, the discretization of which is the same as here, the sequence of approximate velocities satisfy the same estimate as here and the control on the translates of the approximates of ρ being weaker than the control on the sequence $\big(k^{(m)}\big)_{m\in\mathbb{N}}$ obtained here.

Finally, the fact that:

$$\lim_{m\to+\infty} \int_\Omega \lambda(k^{(m)})\ |\boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}|^2\ \pi^{(m)}\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \lambda(k)\ |\boldsymbol{\nabla}_h \boldsymbol{u}|^2\ \varphi \,\mathrm{d}\boldsymbol{x}$$

is an easy consequence of the approximation properties of $\pi^{(m)}$ (combined with the regularity of φ) and the strong convergence in $\mathrm{L}^2(\Omega)^{d\times d}$ of the sequence $(\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ proven in Lemma 3.5.5. □

Gathering all the results of this section, we obtain the following convergence theorem.

Theorem 3.5.8 (Convergence of the scheme)**.** *Let $\big(\mathcal{M}^{(m)}\big)_{m\in\mathbb{N}}$ be a regular sequence of discretizations of Ω, in the sense of Definition 3.2.1. Let $(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of approximate solutions (i.e. , for a given m, $(\boldsymbol{u}^{(m)}, p^{(m)}, k^{(m)})$ is the solution to Problem (3.11) with the mesh $\mathcal{M}^{(m)}$). Then, up to the extraction of a subsequence:*

1. *the sequence $\big(\boldsymbol{u}^{(m)}\big)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^2(\Omega)^d$, to a limit $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$, and the sequence of discrete gradients $\big(\boldsymbol{\nabla}_h \boldsymbol{u}^{(m)}\big)_{m\in\mathbb{N}}$ weakly converges in $\mathrm{L}^2(\Omega)^{d\times d}$ to $\boldsymbol{\nabla}\boldsymbol{u}$,*
2. *$\big(p^{(m)}\big)_{m\in\mathbb{N}}$ weakly converges in $\mathrm{L}^2(\Omega)$ to $p \in \mathrm{L}^2(\Omega)$,*
3. *$\big(k^{(m)}\big)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^\beta(\Omega)$ to $k \in \mathrm{W}^{1,\alpha}_0(\Omega)$, for any $\alpha \in [1, d/(d-1))$ and $\beta \in [1, d/(d-2))$. and the sequence of finite volume gradients $\big(\boldsymbol{\nabla}_{\mathcal{M}} k^{(m)}\big)_{m\in\mathbb{N}}$ weakly converges to $\boldsymbol{\nabla} k$ in $\mathrm{L}^\alpha(\Omega)^d$, for any $\alpha \in [1, d/(d-1))$,*
4. *the sequence $(\lambda(k^{(m)})^{1/2}\ \boldsymbol{\nabla}_h \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ converges (strongly) in $\mathrm{L}^2(\Omega)^{d\times d}$ to $\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u}$,*

and the triplet $(\boldsymbol{u}, p, k)$ is a solution to the weak continuous problem (3.3).

3.6 Towards the extension to a general (unbounded) viscosity

In this section, we prove a stability result for the scheme when the viscosity is unbounded, and we discuss the difficulties posed by the extension to this case of the convergence proof.

To be specific, let us consider that the viscosity is now given, instead of (3.2), by:

$$\lambda(k) = \sqrt{\mu^2 + \ell^2 k} \tag{3.19}$$

To state the weak formulation of the problem, we need to introduce weighted Sobolev spaces, denoted by $\mathrm{W}^{1,p}(\omega;\Omega)$ and defined as follows:

Definition 3.6.1 (Weighted Sobolev spaces)**.** Let ω be a mesurable positive and increasing real function $\omega : \mathbb{R}^+ \to \mathbb{R}^+_\star$, and $\mathrm{W}^{1,p}(\omega;\Omega)$ be the functional space such that:

$$\mathrm{W}^{1,p}(\omega;\Omega) = \big\{ v \in \mathrm{L}^p(\Omega);\ \omega|\boldsymbol{\nabla} v|^p \in \mathrm{L}^1(\Omega) \big\} \tag{3.20}$$

equipped with the following seminorm

$$|u|_{1,p;\omega} = \left(\int_\Omega |\boldsymbol{\nabla} u(\boldsymbol{x})|^p \omega(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{3.21}$$

and the natural norm,

$$\|u\|_{1,p;\omega} = \|u\|_{\mathrm{L}^p(\Omega)} + |u|_{1,p;\omega} \tag{3.22}$$

The definition extends to vector valued functions by the usual way.

By the Cauchy–Schwarz inequality, we remark that, for any regular function u, v:

$$\int_\Omega \omega(\boldsymbol{x}) \boldsymbol{\nabla} u(\boldsymbol{x}) \cdot \boldsymbol{\nabla} v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \le |u|_{1,2;\omega} |v|_{1,2;\omega}, \tag{3.23}$$

On the other hand, since the viscosity $\lambda(k)$ is not bounded, the diffusion term is guaranteed to be finite only for a test function $\boldsymbol{v}$ lying in $\mathrm{W}^{1,\alpha}(\Omega)^d$ for $\alpha > \alpha_0 > 2$, and so the Nečas lemma allows to control the pressure only in $\mathrm{L}^\beta(\Omega)$, with $\beta < \beta_0$ and β_0 being defined by $1/\alpha_0 + 1/\beta_0 = 1$. Let us now compute β_0. Since we search for k in $\mathrm{W}^{1,\gamma}(\Omega)$ with $\gamma < d/(d-1)$, the Sobolev embedding theorem implies that $\lambda(k)^{1/2} \in \mathrm{L}^\zeta(\Omega)$, with $\zeta < 4d/(d-2)$. As a consequence, since the velocity $\boldsymbol{u}$ is sought in $\mathrm{W}^{1,2}_0\big(\lambda;\Omega\big)^d$ (which means that $\lambda(k)^{1/2}\ \boldsymbol{\nabla}\boldsymbol{u} \in \mathrm{L}^2(\Omega)^d$), we get $1/\alpha_0 + 1/2 + (d-2)/4d = 1$, and so $\beta_0 = 4d/(3d-2)$.

Finally, the first two equations of the weak continuous problem may now be reformulated as:

Find $(\boldsymbol{u}, p) \in \mathrm{W}_0^{1,2}\big(\lambda;\Omega\big)^d \times \mathrm{L}^\beta(\Omega)$, with $\beta < 4d/(3d-2)$, such that,
for all $(\boldsymbol{v}, q) \in \mathrm{W}_0^{1,2}\big(\lambda;\Omega\big)^d \times \mathrm{L}^2(\Omega)$:

$$\left| \begin{array}{l} \displaystyle\int_\Omega \lambda(k)\ \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ \displaystyle\int_\Omega q\ \boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = 0, \end{array} \right. \tag{3.24}$$

so that every terms make sense.

Since the discrete functions are finite, the scheme is left unchanged.

To control the discrete pressure, we now need an *inf-sup* relation in $\mathrm{W}^{1,\alpha}(\Omega)$, namely the fact that, for the Crouzeix-Raviart element and any $\beta \in [1, +\infty)$, there exists $c > 0$ independent of the mesh such that:

$$\forall q \in \mathrm{H}_\mathcal{M}, \qquad \sup_{\boldsymbol{v}\in\boldsymbol{V}_h} \frac{\displaystyle\int_\Omega q\ \boldsymbol{\nabla}\cdot_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\displaystyle\Big[\int_\Omega |\boldsymbol{\nabla}_h\boldsymbol{v}|^{\beta'}\,\mathrm{d}\boldsymbol{x}\Big]^{1/\beta'}} =\geq c\ \|q - q_\mathrm{m}\|_{\mathrm{L}^\beta(\Omega)},$$

with $1/\beta + 1/\beta' = 1$. This is an easy consequence of the Nečas lemma and the stability of the projection operator r_h, as stated in Lemma 3.b.1. As as consequence, following the same line as in Theorem 3.4.3, we obtain the following stability result.

Theorem 3.6.2. *Let $\mathcal{M}$ be a discretization of the domain Ω and let $\theta_0 > 0$ be a real number such that $\theta_\mathcal{M} \geq \theta_0$, with $\theta_\mathcal{M}$ defined in* (3.4). *Then a solution $(\boldsymbol{u}, p, k) \in \boldsymbol{V}_h \times \mathrm{H}_\mathcal{M} \times \mathrm{H}_\mathcal{M}$ to the discrete Problem* (3.11) *satisfies the following estimate:*

$$\|\boldsymbol{u}\|_{1,b} + \|k\|_{1,\alpha,\mathcal{M}} + \|p\|_{\mathrm{L}^\beta(\Omega)} \leq C,$$

where α satisfies $1 \leq \alpha < d/(d-1)$, β satisfies $1 \leq \alpha < 4d/(3d-2)$ and C only depends on $\boldsymbol{f}$, Ω, μ, θ_0, α and β.

To prove the convergence of the scheme, all the arguments invoked in the bounded case also hold in the unbounded one, except one difficulty, which we were not able to solve. The problem occurs when trying to establish the strong convergence of the viscous dissipation term, which needs to use the continuous solution $\boldsymbol{u}$ as test function in the weak continuous problem. Replacing the Stokes problem by an elliptic equation, this is performed in [14], by showing that regular functions are dense in the weighted Sobolev space defined by 3.6.1 (Lemma 3.a.2 of the appendix). Unfortunately, this is not sufficient here because of the presence of the pressure term. A way to circumvent this difficulty may be to approach $\boldsymbol{u}$ by a sequence of regular solenoidal functions, but such a result is, at the present time, unknown to us.

3.7 Concluding remarks

In this paper, we have proven the convergence of a numerical approximation of a model problem for RANS modelling of turbulent flows. This model problem couples the Stokes problem with a balance equation for a turbulent scale (often identified to the turbulent energy), the right-hand side of which is the viscous dissipation. As a by-product, this convergence analysis yields an existence result for the solution of the continuous problem (which may also be derived using other arguments [17]).

Several extensions of this work are possible. First, the adaptation of this study to the Rannacher-Turek element, used for hexahedral meshes (together with the Crouzeix-Raviart element for simplices) in the freeware computer code ISIS developped at IRSN [15] on the basis of the software platform PELICANS [18], seems to require only minor modifications; note however that, in this case, the family $(\boldsymbol{x}_K)_{K\in\mathcal{M}}$ is readily built only for structured meshes (*i.e.* rectangles or cuboids), which need to be combined to simplices to deal with general domains. Second, the momentum balance equation may be made more realistic, by adding a convection term (thus passing from Stokes and Navier-Stokes equations), and using the usual form of the diffusion tensor in variable viscosity flows (*i.e.* $\boldsymbol{\tau}(\boldsymbol{u}) = \mu\ (\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u}^t)$. For the first point, the convection term should be designed in order to preserve the stability of the scheme, using for instance the construction presented in [1]. For the second one, since the Crouzeix-Raviart element is known not to enjoy a discrete Korn lemma, a stabilizing term should be added [16, 5]. Provided that these ingredients are used, no additional difficulty seems to be anticipated.

3.a Density of C_c^∞ in a weighted Sobolev space

Let Ω be a open bounded set of $\mathbb{R}^d$ ($d \geq 1$) with a Lipschitz continuous boundary and ω a measurable function from Ω to $\mathbb{R}$ such that $\omega \geq 0$ a.e. in Ω. Let $H_0^1(\omega;\Omega) = \{u \in H_0^1(\Omega),\ \omega Du \in L^2(\Omega)^d\}$, where $Du = (D_1 u, \ldots, D_d u)^t$ and $D_i u$ denotes the (weak) derivative of u with respect to x_i (the variable in $\mathbb{R}^d$ is denoted by $\boldsymbol{x}$ whose components are $x_1, \ldots, x_d$).

The natural norm in the space $H_0^1(\omega;\Omega)$ is given by:

$$\|u\|_{1,\omega}^2 = \|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^{d} \|\omega D_i u\|_{L^2(\Omega)}^2,$$

Where $\|\cdot\|_{L^2(\Omega)}$ stands for the usual $L^2(\Omega)$-norm.

We first remark that if there exists $\alpha > 0$ such that $\omega \geq \alpha$ a.e., then $H_0^1(\omega;\Omega)$ is an Hilbert space. This is given in the following lemma.

Lemma 3.a.1. *Let Ω be a open bounded set of $\mathbb{R}^d$ ($d \geq 1$) with a Lipschitz continuous boundary. Let $\alpha > 0$ and ω be a measurable function from Ω to $\mathbb{R}$ such that $\omega \geq \alpha$ a.e. in Ω. Then, $H_0^1(\omega;\Omega)$ is an Hilbert space.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_0^1(\omega;\Omega)$. Thanks to $\omega \geq \alpha$ a.e., it is also a Cauchy sequence in $H_0^1(\Omega)$. Then there exists $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$ (as $n \to \infty$). Furthermore, for any $i \in \{1, \ldots, d\}$ the sequence $(\omega D_i u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, then it converges to some function, say F_i, in $L^2(\Omega)$. But, since $D_i u_n \to D_i u$ in $L^2(\Omega)$, we can assume, up to a subsequence, that $D_i u_n \to D_i u$ a.e., which gives that $F_i = \omega D_i u$. This proves that $u \in H_0^1(\omega;\Omega)$ and $u_n \to u$ in $H_0^1(\omega;\Omega)$ as $n \to \infty$.

The next question is the embedding of $C_c^\infty(\Omega)$ in $H_0^1(\omega;\Omega)$. Actually, even if $d = 1$ and $\Omega = (0,1)$, it is possible to have $C_c^\infty(\Omega) \cap H_0^1(\omega;\Omega) = \{0\}$ with a measurable function ω such that $\omega \geq 1$ a.e. in Ω. But, if $\omega \in L^2_{loc}(\Omega)$, then one has $C_c^\infty(\Omega) \subset H_0^1(\omega;\Omega)$ (since, for $\varphi \in C_c^\infty(\Omega)$, one has, for any $i \in \{1, \ldots, d\}$, $D_i\varphi \in L^\infty(\Omega)$ and $D_i\varphi$ has a compact support in Ω which give $\omega D_i \varphi \in L^2(\Omega)$).

Now assuming that $\omega \in L^2_{loc}(\Omega)$ (in order to have $C_c^\infty(\Omega) \subset H_0^1(\omega;\Omega)$) and that there exists some $\alpha > 0$ such that $\omega \geq \alpha$ a.e. in Ω (which gives that $H_0^1(\omega;\Omega)$ is an Hilbert space), do we have the density of $C_c^\infty(\Omega)$ in $H_0^1(\omega;\Omega)$? The answer is obviously "yes" if $\omega \in L^\infty(\Omega)$ (since, in this case, $H_0^1(\omega;\Omega) = H_0^1(\Omega)$ and the norm in $H_0^1(\omega;\Omega)$ is equivalent to the usual norm in $H_0^1(\Omega)$). We will prove in the next lemma that the answer is also "yes" if $\omega \in H^1(\Omega)$, a first proof of this result can be found in [14], we give it here for the sake of completness. ☐

Lemma 3.a.2. *Let Ω be a open bounded set of $\mathbb{R}^d$ ($d \geq 1$) with a Lipschitz continuous boundary. Let $\alpha > 0$ and $\omega \in H^1(\Omega)$ such that $\omega \geq \alpha$ a.e. in Ω. Then, $C_c^\infty(\Omega) \subset H_0^1(\omega;\Omega)$ and $C_c^\infty(\Omega)$ is dense $H_0^1(\omega;\Omega)$.*

Proof. We already remark (using the fact that $\omega \in L^2_{loc}(\Omega)$) that $C_c^\infty(\Omega) \subset H_0^1(\omega;\Omega)$. We now prove the density of $C_c^\infty(\Omega)$ in $H_0^1(\omega;\Omega)$ in 3 steps.

Step 1. We prove, in this step, that $H_0^1(\omega;\Omega) \cap L^\infty(\Omega)$ is dense in $H_0^1(\omega;\Omega)$. For this step, we only use the fact that $\omega \in L^2(\Omega)$. For $n \in \mathbb{N}$, we define the function T_n from $\mathbb{R}$ to $\mathbb{R}$ by $T_n(s) = \max(\min(s,n), -n)$. Let $u \in H_0^1(\omega;\Omega)$ and, for $n \in \mathbb{N}$, $u_n = T_n(u)$. The Dominated Convergence Theorem gives $u_n \to u$ in $L^2(\Omega)$, as $n \to \infty$. Furthermore, a classical Stampacchia result gives that $u_n \in H_0^1(\Omega)$ and, for any $i \in \{1, \ldots, d\}$, $D_i(u_n) = (D_i u)1_{|u|\leq n}$. With this equality one deduces that $u \in H_0^1(\omega;\Omega)$ and, applying once again the Dominated Convergence Theorem, one obtain the convergence of u_n to u in $H_0^1(\omega;\Omega)$. Since $u_n \in H_0^1(\omega;\Omega) \cap L^\infty(\Omega)$, this concludes the first step.

Step 2. We prove, in this step, that $H_0^1(\omega;\Omega) \cap L_c^\infty(\Omega)$ is dense in $H_0^1(\omega;\Omega)$, where $L_c^\infty(\Omega)$ denotes the elements of $L^\infty(\Omega$ with compact support (that is to say that $v \in L_c^\infty(\Omega)$ if $v \in L^\infty(\Omega)$ and there exists a compact subset of Ω, K, such that $v = 0$ a.e. in $\Omega \setminus K$). For this step, we use the fact that $\omega \in H^1(\Omega)$.

Thanks to the regularity of the boundary of Ω, it is possible to construct an increasing sequence $(\Omega_n)_{n\in\mathbb{N}^\star}$ of open subset of Ω and a sequence $(\varphi_n)_{n\in\mathbb{N}^\star} \subset C_c^\infty(\Omega)$ such that, for all $n \in \mathbb{N}^\star$:
- $\varphi_n = 1$ in Ω_n, $\varphi_n = 0$ in Ω_{2n}, $|\varphi_n| \leq 1$ in Ω,
- $\mathrm{dist}(\Omega_n, \Omega^c) \leq 1/n$,

– $|D_i\varphi_n| \le Cn$, for all $i \in \{1,\dots,d\}$, where C only depends on Ω.

Let $u \in \mathrm{H}^1_0(\omega;\Omega) \cap \mathrm{L}^\infty(\Omega)$. For $n \in \mathbb{N}^\star$, we set $u_n = u\varphi_n$ so that $u_n \in \mathrm{L}^\infty_c(\Omega)$ and $u_n \to u$ in $\mathrm{L}^2(\Omega)$, as $n \to \infty$ (thanks to the Dominated Convergence Theorem). We now have to prove that $u \in \mathrm{H}^1_0(\omega;\Omega)$ and $u_n \to u$ in $\mathrm{H}^1_0(\omega;\Omega)$, as $n \to \infty$.

Let $i \in \{1,\dots,d\}$. We have $\omega D_i u_n = \omega(D_i u)\varphi_n + \omega u D_i\varphi_n$. Since $\omega D_i u \in \mathrm{L}^2(\Omega)$, $\omega \in \mathrm{L}^2(\Omega)$ and φ_n, $D_i\varphi_n$, $u \in \mathrm{L}^\infty$, one has $u \in \mathrm{H}^1_0(\omega;\Omega)$. Furthermore, since $\omega D_i u \in \mathrm{L}^2(\Omega)$, the Dominated Convergence Theorem gives $\omega(D_i u)\varphi_n \to \omega D_i u$, as $n \to \infty$. In order to conclude that $u_n \to u$ in $\mathrm{H}^1_0(\omega;\Omega)$, we only have to prove that $\omega u D_i\varphi_n$ tends to 0 in $\mathrm{L}^2(\Omega)$ (as $n \to \infty$). To prove this convergence, let us assume for a moment that $\omega u \in \mathrm{H}^1_0(\Omega)$. Then, the Hardy Inequality gives that $(\omega u)/\delta \in \mathrm{L}^2(\Omega)$ (where $\delta(\boldsymbol{x})$ is, for $x \in \Omega$, the distance from x to the boundary of Ω) from which one deduces:

$$\|\omega u D_i\varphi_n\|^2_\delta \le C^2 \int_{\Omega\setminus\Omega_n} \frac{\omega(x)^2 u(\boldsymbol{x})^2}{\delta(\boldsymbol{x})^2} dx \to 0, \text{ as } n \to \infty,$$

since the measure of $\Omega \setminus \Omega_n$ tends to 0 as $n \to \infty$. We have prove that $\omega D_i u_n \to \omega D_i u$ in $\mathrm{L}^2(\Omega)$, as $n \to \infty$ and this proves that $u_n \to u$ $\mathrm{H}^1_0(\omega;\Omega)$. This gives (using also Step 1) the desired density result, namely the density of $\mathrm{H}^1_0(\omega;\Omega) \cap \mathrm{L}^\infty_c(\Omega)$ in $\mathrm{H}^1_0(\omega;\Omega)$.

To conclude this step, one has to prove that $\omega u \in \mathrm{H}^1_0(\Omega)$. This is consequence of $\omega \in \mathrm{H}^1(\Omega)$ and $u \in \mathrm{H}^1_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ as we now show. Since $\omega \in \mathrm{H}^1(\Omega)$ and $u \in \mathrm{H}^1_0(\Omega)$, there exists two sequences $(\psi_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty(\mathbb{R}^d)$ and $(\xi_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\Omega)$ such that $\psi_n \to \omega$ in $\mathrm{H}^1(\Omega)$ and $\xi_n \to u$ in $\mathrm{H}^1_0(\Omega)$ as $n \to \infty$. Then we have $\psi_n\xi_n \in \mathrm{C}^\infty_c(\Omega)$, $\psi_n\xi_n \to \omega u$ in $\mathrm{L}^1(\Omega)$ and, for any $i \in \{1,\dots,d\}$, $D_i(\psi_n\xi_n) = D_i(\psi_n)\xi_n + \psi_n D_i\xi_n \to D_i(\omega)u + \omega D_i u$ in $\mathrm{L}^1(\Omega)$ as $n \to \infty$. This proves that $\omega u \in \mathrm{W}^{1,1}_0(\Omega)$ and $D_i(\omega u) = D_i(\omega)u + \omega D_i u$ a.e..

We now use the fact that $u \in \mathrm{L}^\infty(\Omega)$ and $u \in \mathrm{H}^1_0(\omega;\Omega)$. It gives (thanks to $\omega \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{L}^\infty(\Omega)$) that $\omega u \in \mathrm{L}^2(\Omega)$ and (thanks to $D_i\omega \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{H}^1_0(\omega;\Omega) \cap \mathrm{L}^\infty(\Omega)$) that $D_i(\omega u) \in \mathrm{L}^2(\Omega)$ for any $i \in \{1,\dots,d\}$. Then, one has $\omega u \in \mathrm{H}^1(\Omega)$. Finally, since we already know $\omega u \in \mathrm{W}^{1,1}_0(\Omega)$, the trace of ωu on the boundary of Ω is zero and this gives, as we claimed, that $\omega u \in \mathrm{H}^1_0(\Omega)$.

Step 3. In this last step we prove the density of $\mathrm{C}^\infty_c(\Omega)$ in $\mathrm{H}^1_0(\omega;\Omega)$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers, that is $r_n(\boldsymbol{x}) = n^d r(nx)$, for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}^\star$, with $r \in \mathrm{C}^\infty_c(\mathbb{R}^d)$, $r \ge 0$ in $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} r(\boldsymbol{x})dx = 1$.

Let $u \in \mathrm{H}^1_0(\omega;\Omega) \cap \mathrm{L}^\infty_c(\Omega)$, we set, for all $n \in \mathbb{N}^\star$, $u_n = u \star r_n$ (we define u in the whole $\mathbb{R}^d$ by taking $u = 0$ in Ω^c). For all $n \in \mathbb{N}^\star$, $u_n \in C^\infty_c(\mathbb{R}^d)$ and the restriction of u_n to Ω belongs to $\mathrm{C}^\infty_c(\Omega)$ for n great enough (since u has a compact support in Ω, up to negligible set). For simplicity in the proof below, we can assume (skipping if necessary the first terms of the sequence $(u_n)_{n\in\mathbb{N}}$) that $(u_n)_{n\in\mathbb{N}^\star} \subset \mathrm{C}^\infty_c(\Omega)$.

In order to conclude our proof of density, it will be sufficient to prove that $u_n \to u$ in $\mathrm{H}^1_0(\omega;\Omega)$, as $n \to \infty$. However, this convergence seems not easy to prove (and, by the way, is perhaps not true). We will prove below that $v_n \to u$ in $\mathrm{H}^1_0(\omega;\Omega)$, as $n \to \infty$, where v_n is a (convenient) finite convex combination of $\{u_q,\ q \ge n\}$. Since v_n also belongs to $\mathrm{C}^\infty_c(\Omega)$ for all $n \in \mathbb{N}^\star$, the proof of Lemma 3.a.2 is then complete.

We first remark that, since $u \in \mathrm{L}^2(\mathbb{R}^d)$, one has $u_n \to u$ in $\mathrm{L}^2(\mathbb{R}^d)$ (and therefore in $\mathrm{L}^2(\Omega)$), as $n \to \infty$.

Let $i \in \{1,\dots,d\}$, it seems not easy to prove that $\omega D_i u_n \to \omega D_i u$ in $\mathrm{L}^2(\Omega)$, as $n \to \infty$. We will only prove that the sequence $(\omega D_i u_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^2(\Omega)$. For simplicity in the proof below, we will define ω in Ω^c in such a way that $\omega \in \mathrm{H}^1(\mathbb{R}^d)$, this is possible thanks to the regularity of the boundary of Ω. (Indeed, the definition of ω in Ω^c is unuseful since there exists a subset K of Ω such that $u = u_n = 0$ a.e. in K^c, at least for n great enough.)

One has $\omega D_i u_n = \omega(u \star D_i r_n) = (D_i(\omega u)) \star r_n + R_n$, with $R_n = \omega(u \star D_i r_n) - (\omega u) \star D_i r_n$. Since $\omega u \in \mathrm{H}^1(\mathbb{R}^n)$ (as we show in Step 2), one has $(D_i(\omega u)) \star r_n \to D_i(\omega u)$ in $\mathrm{L}^2(\mathbb{R}^d)$, as $n \to \infty$, and therefore the sequence $((D_i(\omega u)) \star r_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^2(\mathbb{R}^d)$. We are now seeking for a bound of the L^2-norm of R_n. One has, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} R_n(\boldsymbol{x}) = \int_{\mathbb{R}^d} (\omega(x) - \omega(x-y))u(x-y)D_i r_n(y)dy &= \int_{\mathbb{R}^d} (\omega(x) - \omega(x-y))u(x-y)nD_i r(ny)dy \\ &= \int_{\mathbb{R}^d} (\omega(x) - \omega(x - \frac{z}{n}))u(x - \frac{z}{n})nD_i r(z)dz. \end{aligned}$$

Let $a > 0$ such that $r = 0$ in B_a^c where $B_a = \{x \in \mathbb{R}^d,\ |x| \le a\}$ and let $M = \|u\|_\infty \|D_i r\|_\infty$. We have, for all $x \in \mathbb{R}^d$,

$$R_n(\boldsymbol{x})^2 \le M^2 n^2 \mathrm{meas}(B_a) \int_{B_a} (\omega(x) - \omega(x - \frac{z}{n}))^2 dz,$$

where $\mathrm{meas}(B_a)$ denotes the Lebesgue measure of B_a. Integrating with respect to x (and using the Fubini-Tonelli's Theorem) leads to

$$\|R_n\|_{\mathrm{L}^2(\Omega)} \le Mn\,\mathrm{meas}(B_a) \sup_{z\in B_a} \|\omega(\cdot) - \omega(\cdot - \frac{z}{n})\|_{\mathrm{L}^2(\Omega)}.$$

But, it is well known that $\|\omega(\cdot) - \omega(\cdot + h)\|_2 \le \|\omega\|_{\mathrm{H}^1(\mathbb{R}^d)}|h|$ for all $h \in \mathbb{R}^d$ (see, for instance, Lemma B5 of [12]). Then, we have $\|R_n\|_2 \le Ma\,\mathrm{meas}(B_a)\|\omega\|_{\mathrm{H}^1(\mathbb{R}^d)}$. This proves that the sequence $(R_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^2(\mathbb{R}^d)$.

Thanks to the bound on $\|R_n\|_2$, The sequence $(u_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{H}_0^1(\omega;\Omega)$. Since $\mathrm{H}_0^1(\omega;\Omega)$ is an Hilbert space, there exists $v \in \mathrm{H}_0^1(\omega;\Omega)$ and there exists a subsequence, still denoted $(u_n)_{n\in\mathbb{N}^\star}$, such that $u_n \to v$ weakly in $\mathrm{H}_0^1(\omega;\Omega)$, as $n \to \infty$. Now using the Mazur's Lemma, there exists, for all $n \in \mathbb{N}^\star$ v_n, finite convex combination of $\{u_q,\ q \ge n\}$, such that $v_n \to v$ (strongly) in $\mathrm{H}_0^1(\omega;\Omega)$, as $n \to \infty$. In particular (since $\|w\|_2 \le \|w\|_{1,\omega}$ for all $w \in \mathrm{H}_0^1(\omega;\Omega)$), one has $v_n \to v$ in $\mathrm{L}^2(\Omega)$, as $n \to \infty$. But, we already know that $u_n \to u$ in $\mathrm{L}^2(\Omega)$, then it is easy to show that $v_n \to u$ in $\mathrm{L}^2(\Omega)$ (since v_n is a finite convex combination of $\{u_q,\ q \ge n\}$). Therefore, one has $u = v$ a.e., which gives that $v_n \to u$ in $\mathrm{H}_0^1(\omega;\Omega)$, as $n \to \infty$, and concludes the proof of Lemma 3.a.2 (since $v_n \in \mathrm{C}_c^\infty(\Omega)$ for all $n \in \mathbb{N}^\star$). □

3.b L^p stability of the interpolation operator

We prove here an L^p stability result of the projectors from $\mathrm{W}_0^{1,p}(\Omega)$ to the discrete space of Crouzeix-Raviart functions on a classical discretization of Ω using triangles, if $d = 2$, or tetrahedra, if $d = 3$ (actually, it is also possible to work with the Rannacher-Turek element). The projector is defined as follows. Let $p \in [1,\infty)$, for $u \in \mathrm{W}_0^{1,p}(\Omega)$, one defines $\Pi(u)$ as the unique function in the discrete space satisfying

$$\int_\sigma \Pi(u)(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_\sigma u(\boldsymbol{x}) d\gamma(\boldsymbol{x}),$$

for all edge (if $d = 2$) or interface (if $d = 3$) on the mesh. (The variable in $\mathbb{R}^d$ is denoted by $\boldsymbol{x}$ and γ is the $(d-1)$-Lebesgue measure on σ.) The norm in the discrete space is the usual $\mathrm{W}_0^{1,p}$-broken norm denoted in Lemma 3.b.1 as $\|\cdot\|_{1,p}$.

Lemma 3.b.1 (Stability for the Crouzeix-Raviart element)**.**
Let $u \in \mathrm{W}_0^{1,p}(\Omega)$, then:

$$\|\Pi(u)\|_{1,p} \le d^{\frac{1}{p}+\frac{1}{2}} \|u\|_{\mathrm{W}_0^{1,p}(\Omega)}. \tag{3.25}$$

Proof. Lemma 3.b.1 We recall that $\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} = \int_\Omega |\boldsymbol{\nabla} u|^p\, \mathrm{d}\boldsymbol{x}$ and $\|v\|^p_{1,p} = \sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla} u|^p\, \mathrm{d}\boldsymbol{x}$, where $\mathcal{M}$ denotes the set of the elements of the mesh.

Let $u \in \mathrm{C}^\infty(\mathbb{R}^d)$ and K be an element of the mesh. We set $v = \Pi(u)$. Since $\partial v/\partial x$ is a constant function over K, one has

$$\int_K \left|\frac{\partial v}{\partial x}\right|^p \mathrm{d}\boldsymbol{x} = |K| \left|\frac{\partial v}{\partial x}\right|^p = |K|^{1-p} \left|\int_K \frac{\partial v}{\partial x}\,\mathrm{d}\boldsymbol{x}\right|^p.$$

Integrating by part, we have

$$\int_K \frac{\partial v}{\partial x}\,\mathrm{d}\boldsymbol{x} = \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma v n_x\,\mathrm{d}\boldsymbol{x} = \sum_{\sigma\in\mathcal{E}(K)} \left(\int_\sigma v\,\mathrm{d}\boldsymbol{x}\right) n_x,$$

where n_x is the first component of the normal vector to ∂K exterior to K and $\mathcal{E}(K)$ denotes the three edges, if $d = 2$, or the fourth interfaces, if $d = 3$, of K.

Since $\left(\int_\sigma v \,\mathrm{d}\boldsymbol{x}\right) = \left(\int_\sigma u \,\mathrm{d}\boldsymbol{x}\right)$, we then have

$$\int_K \frac{\partial v}{\partial x} \,\mathrm{d}\boldsymbol{x} = \sum_{\sigma \in \mathcal{E}(K)} \left(\int_\sigma u \,\mathrm{d}\boldsymbol{x}\right) n_x = \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma u n_x \,\mathrm{d}\boldsymbol{x} = \int_K \frac{\partial u}{\partial x} \,\mathrm{d}\boldsymbol{x},$$

and therefore

$$\int_K \left|\frac{\partial v}{\partial x}\right|^p \,\mathrm{d}\boldsymbol{x} = |K|^{1-p} \left|\int_K \frac{\partial u}{\partial x} \,\mathrm{d}\boldsymbol{x}\right|^p.$$

We now use the Hölder Inequality, it yields

$$\left|\int_K \frac{\partial u}{\partial x} \,\mathrm{d}\boldsymbol{x}\right|^p \leq |K|^{p-1} \int_K \left|\frac{\partial u}{\partial x}\right|^p \,\mathrm{d}\boldsymbol{x}.$$

We then obtain

$$\int_K \left|\frac{\partial v}{\partial x}\right|^p \,\mathrm{d}\boldsymbol{x} \leq \int_K \left|\frac{\partial u}{\partial x}\right|^p \,\mathrm{d}\boldsymbol{x} \leq \int_K |\boldsymbol{\nabla} u|^p \,\mathrm{d}\boldsymbol{x}.$$

Using a similar proof for the other derivatives, we obtain

$$\int_K |\boldsymbol{\nabla} v|^p \,\mathrm{d}\boldsymbol{x} \leq d^{1+\frac{p}{2}} \int_K |\boldsymbol{\nabla} u|^p \,\mathrm{d}\boldsymbol{x}.$$

Summing over all the elements of the mesh, we obtain Inequality 3.25 when $u \in \mathrm{C}_c^\infty(\Omega)$. By density of $\mathrm{C}^\infty(\mathbb{R}^d)$ in $\mathrm{W}^{1,p}(\Omega)$, we obtain Inequality 3.25 for $u \in \mathrm{W}_0^{1,p}(\Omega)$ (and even for $u \in \mathrm{C}^\infty(\mathbb{R}^d)$ or $u \in \mathrm{W}^{1,p}(\Omega)$, but in this case the left and right hand sides of Inequality 3.25 are only semi-norms). □

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Chapitre 4

Convergence of a finite volume scheme for the convection–diffusion equation with L^1 data

CONVERGENCE OF A FINITE VOLUME SCHEME FOR THE CONVECTION-DIFFUSION EQUATION WITH L^1 DATA

Abstract. In this paper, we prove the convergence of a finite-volume scheme for the time-dependent convection–diffusion equation with an L^1 right-hand side. To this purpose, we first prove estimates for the discrete solution and for its discrete time-derivative. Then we show the convergence of a sequence of discrete solutions obtained with more and more refined discretizations, possibly up to the extraction of a subsequence, to a function which mets the regularity requirements of the weak formolulation of the problem ; to this purpose, we prove a compactness result, which may be seen as a discrete analogue to Aubin-Simon's lemma. Finally, such a limit is shown be indeed a weak solution.

4.1 Introduction

We address in this paper the discretization by a finite volume method of the following problem:

$$\begin{aligned} &\partial_t u + \boldsymbol{\nabla}\cdot(u\,\boldsymbol{v}) - \Delta u = f && \text{in } \Omega \times (0,T), \\ &u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) && \text{a.e. in } \Omega, \\ &u(\boldsymbol{x},t) = 0 && \text{a.e. in } \partial\Omega \times (0,T), \end{aligned} \tag{4.1}$$

where Ω is an open, bounded, connected subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, which is supposed to be polygonal ($d = 2$) or polyhedral ($d = 3$), and $\partial\Omega$ stands for its boundary. The right-hand side, f, and the initial condition, u_0, are supposed to satisfy:

$$f \in L^1(\Omega \times (0,T)), \qquad u_0 \in L^1(\Omega). \tag{4.2}$$

The vector-valued velocity field $\boldsymbol{v}$ is supposed to be divergence-free, to vanish on the boundary of the domain and to be regular, let us say:

$$\begin{aligned} &\boldsymbol{v} \in C^1(\bar{\Omega} \times [0,T]), \\ &\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t) = 0,\ \forall(\boldsymbol{x},t) \in \Omega \times (0,T), \\ &\boldsymbol{v}(\boldsymbol{x},t) = 0,\ \forall(\boldsymbol{x},t) \in \partial\Omega \times (0,T). \end{aligned} \tag{4.3}$$

Definition 4.1.1 (Weak solution)**.** We define a weak solution u to problem (4.1)-(4.2) by:

$$u \in \cup_{1 \le q < (d+2)/(d+1)}\ L^q(0,T; W_0^{1,q}(\Omega))$$

and, $\forall \varphi \in C_c^\infty(\Omega \times [0,T))$:

$$\begin{aligned} &-\int_{\Omega\times(0,T)} u(\boldsymbol{x},t)\,\partial_t\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\Omega\times(0,T)} u\,\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+\int_{\Omega\times(0,T)} \nabla u(\boldsymbol{x},t)\cdot\nabla\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{\Omega\times(0,T)} f\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega u_0(\boldsymbol{x})\,\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

The existence of such a weak solution (and to more general nonlinear elliptic and parabolic problems) has been proven in [4]; further developments (in particular, concerning the uniqueness of solutions) can be found in [1, 2, 3, 18, 10].

The motivation of this study lies in the fact that Problem (4.1)-(4.3) is a model problem for a class of convection-diffusion-reaction equations with L^1–data encountered in the so-called *Reynolds Averaged Navier–Stokes* (RANS) modeling of turbulent flows. In this class of models, the effects of turbulent stresses are taken into account by an additional diffusion term in the momentum balance equation of the averaged Navier–Stokes system governing the evolution of the mean velocity field $\bar{\boldsymbol{v}}$ and pressure $\bar{p}$. This system of equations is given here for reference, in the case of incompressible flows, with μ the laminar viscosity and $\mathbf{g}$ a forcing-term:

$$\begin{aligned} &\partial_t \bar{\boldsymbol{v}} + \boldsymbol{\nabla}\cdot(\bar{\boldsymbol{v}} \otimes \bar{\boldsymbol{v}}) - \boldsymbol{\nabla}\cdot\Big(\big(\mu + \mu_t(\cdot)\big)\nabla \bar{\boldsymbol{v}}\Big) + \nabla \bar{p} = \mathbf{g}, \\ &\boldsymbol{\nabla}\cdot \bar{\boldsymbol{v}} = 0. \end{aligned} \tag{4.4}$$

The additional diffusivity μ_t, called "turbulent viscosity", needs to be modelled by an algebraic relation. Usually, this relation involves a set of characteristic turbulent scales $(\chi_i)_{0<i\leq n}$; for instance the turbulent kinetic energy k (m^2/s^2), its dissipation rate ε (m^2/s^3) or the turbulent frequency scale ω (s^{-1}) are used in two–equation models like the $k-\varepsilon$ model of Launder–Spalding and the $k-\omega$ model of Wilcox. Turbulent scales themselves have to be computed by solving a set of scalar convection–diffusion equations, commonly called "turbulent transport equations", which share the same structure:

$$\partial_t \chi_i + \boldsymbol{\nabla}\cdot(\chi_i \bar{\boldsymbol{v}}) - \boldsymbol{\nabla}\cdot\Big(\lambda\big(\{\chi_p\}_{0<p\leq n}\big)\nabla \chi_i\Big) = f_{\chi_i}(\{\chi_p\}_{0<p\leq n}). \tag{4.5}$$

In these equations, following from the Boussinesq hypothesis, every source term f_{χ_i} is linear (with a bounded coefficient) with respect to $|\nabla \bar{\boldsymbol{v}}|^2 = \sum_{i,j=1}^{d} (\partial_j \bar{v}_i)^2$, with $\bar{v}_i$ the i–th component of $\bar{\boldsymbol{v}}$. Since $\bar{\boldsymbol{v}}$ satisfies the classical energy estimate of the Navier–Stokes analysis, which can be derived from System (4.4), $\nabla \bar{\boldsymbol{v}}$ belongs to $L^2(\Omega \times (0,T))$, and the right-hand side of Equation (4.5) lies in $L^1(\Omega \times (0,T))$.

Let us mention that convection-diffusion equations with L^1 data are also encountered in electrodynamics modeling [6] or heating by induction [7].

In this paper, we show that a sequence of approximate solutions obtained by a backward-in-time and upwind finite volume method converges up to a subsequence towards a function $\bar{u}$ which is a weak solution of the problem, in the sense of Definition 4.1.1. To this purpose, we extend to the time-dependent case techniques developed for steady problems with L^1 data, for a single elliptic equation in [14, 9] and for a system of two coupled elliptic equations arising in heat dissipation by the Joule effect in [6].

The presentation is organized as follows. In Section 4.2, we define the approximation spaces and describe the discrete functional analysis framework which is used in the subsequent developments. Then we prove an abstract compactness result, which may be considered as a discrete analogue of the classical Aubin-Simon's lemma (Section 4.3). The scheme is then given (Section 4.4), then we derive estimates satisfied by the discrete solution (Section 4.5), and conclude by the convergence analysis (Section 4.6).

4.2 Discrete spaces and functional framework

An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

1. $\mathcal{M}$ is a finite family of non empty open polygonal ($d=2$) or polyhedral ($d=3$) convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K.
2. $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "faces" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^d$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is a non empty open subset of E. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}(K)$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}(K)} \overline{\sigma}$. It results from the previous hypotheses that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K,L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$. We denote by $\mathcal{E}_{\rm ext}$ the set of the faces included in $\partial\Omega$ and $\mathcal{E}_{\rm int} = \mathcal{E} \setminus \mathcal{E}_{\rm ext}$ the set of internal faces.
3. $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$. The family $\mathcal{P}$ is supposed to be such that, for all $K \in \mathcal{M}$, $\boldsymbol{x}_K \in K$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K,L) \in \mathcal{M}^2$ with $\sigma = K|L$, it is assumed that the straight line $(\boldsymbol{x}_K, \boldsymbol{x}_L)$ going through $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ is orthogonal to $K|L$.

By $|K|$ and $|\sigma|$, we denote hereafter respectively the measure of the control volume K and of the face σ. For any control volume K and face σ of K, we denote by $d_{K,\sigma}$ the Euclidean distance between $\boldsymbol{x}_K$ and σ and by $\boldsymbol{n}_{K,\sigma}$ the unit vector normal to σ outward from K. For any face σ, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, if σ separates the two control volumes K and L (in which case d_σ is the Euclidean distance between $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$) and $d_\sigma = d_{K,\sigma}$ if σ is included in the boundary. For any control volume $K \in \mathcal{M}$, h_K stands for the diameter of K. We denote by $h_\mathcal{M}$ the quantity $h_\mathcal{M} = \max_{K\in\mathcal{M}} h_K$.

Let $\mathrm{H}_\mathcal{M} \subset \mathrm{L}^\infty(\Omega)$ be the space of functions piecewise constant over any element $K \in \mathcal{M}$. For a finite $q \geq 1$, we define a discrete $\mathrm{W}_0^{1,q}$–norm by:

$$\|u\|_{1,q,\mathcal{M}}^q = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L} |\sigma|\, d_\sigma \left|\frac{u_K - u_L}{d_\sigma}\right|^q + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}},\ \sigma\in\mathcal{E}(K)} |\sigma|\, d_\sigma \left|\frac{u_K}{d_\sigma}\right|^q.$$

We also define:

$$\begin{aligned}\|u\|_{1,\infty,\mathcal{M}} = \quad & \max \Big\{ \big\{ \frac{|u_K - u_L|}{d_\sigma},\ \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L \big\} \\ & \cup \big\{ \frac{|u_K|}{d_\sigma},\ \sigma \in \mathcal{E}_{\mathrm{ext}},\ \sigma \in \mathcal{E}(K) \big\} \ \cup\ \big\{ |u_K|,\ K \in \mathcal{M} \big\} \Big\}\end{aligned}$$

For $q > 1$, we associate to this norm a dual norm with respect to the L^2 inner product, denoted by $\|\cdot\|_{-1,q',\mathcal{M}}$ with q' given by $1/q + 1/q' = 1$ if q is finite and $q' = 1$ if $q = +\infty$, and defined by:

$$\|u\|_{-1,q',\mathcal{M}} = \sup_{v\in\mathrm{H}_\mathcal{M}(\Omega),\ v\neq 0} \quad \frac{1}{\|v\|_{1,q,\mathcal{M}}} \int_\Omega uv \,\mathrm{d}\boldsymbol{x}.$$

As a consequence of the discrete Hölder inequality, the following bound holds for any $q,\ r \in [1, +\infty)$ such that $q < r$:

$$\forall u \in \mathrm{H}_\mathcal{M}, \qquad \|u\|_{1,q,\mathcal{M}} \leq (d\,|\Omega|)^{1/q-1/r}\ \|u\|_{1,r,\mathcal{M}}, \tag{4.6}$$

and, consequently, for any $q,\ r \in [1, +\infty)$ such that $q < r$:

$$\forall u \in \mathrm{H}_\mathcal{M}, \qquad \|u\|_{-1,q,\mathcal{M}} \leq (d\,|\Omega|)^{1/q-1/r}\ \|u\|_{-1,r,\mathcal{M}}. \tag{4.7}$$

We denote by $\xi_\mathcal{M} > 0$ a positive real number such that:

$$\forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}(K), \quad \xi_\mathcal{M} \leq \frac{d_{K,\sigma}}{d_\sigma}, \quad \text{and} \quad \xi_\mathcal{M} \leq \frac{d_{K,\sigma}}{h_K}. \tag{4.8}$$

The greatest real number satisfying these inequalities may be considered as a measure of the regularity of the mesh.

The following discrete Sobolev inequalities are proven in [11, Lemma 9.5, p.790] and [8, 12].

Lemma 4.2.1 (Discrete Sobolev inequality)**.** *For any $q \in [1, d)$, there exists a real number $C(\Omega, \xi_\mathcal{M}, q) > 0$ such that:*

$$\|u\|_{\mathrm{L}^{q^\star}(\Omega)} \leq C(\Omega, \xi_\mathcal{M}, q)\ \|u\|_{1,q,\mathcal{M}} \qquad \textit{with} \quad q^\star = \frac{d\,q}{d-q}.$$

For $q \geq d$ and any $p \in [1, +\infty)$, there exists a real number $C(\Omega, \xi_\mathcal{M}, p) > 0$ such that:

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C(\Omega, \xi_\mathcal{M}, p)\ \|u\|_{1,q,\mathcal{M}}.$$

In addition, the following bound is proven in [12, Lemma 5.4]

Lemma 4.2.2 (Space translates estimates)**.** *Let $v \in \mathrm{H}_\mathcal{M}$, and let $\bar{v}$ be its extension by 0 to $\mathbb{R}^d$. Then:*

$$\|\bar{v}(\cdot + \boldsymbol{y}) - \bar{v}(\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)} \leq \sqrt{d}\ |\boldsymbol{y}|\ \|v\|_{1,1,\mathcal{M}}, \quad \forall \boldsymbol{y} \in \mathbb{R}^d.$$

The following result is a consequence of the Kolmogorov's theorem and of this inequality.

Theorem 4.2.3. *Let $\mathcal{M}^k$ be a sequence of meshes the step of which tends to zero, and regular in the sense that any $\mathcal{M}^k$, $k \in \mathbb{N}$, satisfies the regularity assumption* (4.8) *with a unique (i.e. independent of k) positive real number ξ.*
Let $q \in [1, +\infty)$, and let $(u^k_{\mathcal{M}})_{k\in\mathbb{N}}$ be a sequence of discrete functions (i.e. such that, $\forall k \in \mathbb{N}$, $u^k \in \mathrm{H}^k_{\mathcal{M}}$, where $\mathrm{H}^k_{\mathcal{M}}$ is the discrete space associated to $\mathcal{M}^k$) satisfying:

$$\forall k \in \mathbb{N}, \qquad \|u^k\|_{1,q,\mathcal{M}} \leq C$$

where C is a given positive real number. Then, possibly up to the extraction of a subsequence, the sequence $(u^k_{\mathcal{M}})_{k\in\mathbb{N}}$ converges in $\mathrm{L}^p(\Omega)$ to a function $u \in \mathrm{W}^q_0(\Omega)$, for any $p \in [1, q^\star)$, where $q^\star = d\,q/(d-q)$ if $q < d$ and $q^\star = +\infty$ otherwise.

Furthermore, we suppose given a uniform partition of the time-interval $[0,T)$, such that $[0,T] = \cup_{0\leq n<N}[t^n, t^{n+1}]$ (so $t^n = n\,\delta t$, with $\delta t = T/N$). Let $\mathrm{H}_{\mathcal{D}}$ be the space of piecewise constant functions over each $K \times I^n$, for $K \in \mathcal{M}$ and $I^n = (t^n,\ t^{n+1})$, $0 \leq n < N$. To each sequence $(u^n)_{n=0,N}$ of functions of $\mathrm{H}_{\mathcal{M}}(\Omega)$, we associate the function $u \in \mathrm{H}_{\mathcal{D}}$ defined by:

$$u(\boldsymbol{x}, t) = u^{n+1}(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \Omega, \text{ for any } t \in (t^n, t^{n+1}),\ 0 \leq n < N. \tag{4.9}$$

In addition, for any $u \in \mathrm{H}_{\mathcal{D}}$ and $1 \leq n \leq N$, we define $\partial_{t,\mathcal{D}}(u)^n \in \mathrm{H}_{\mathcal{M}}$ by:

$$\partial_{t,\mathcal{D}}(u)^n(\boldsymbol{x}) = \frac{u^n(\boldsymbol{x}) - u^{n-1}(\boldsymbol{x})}{\delta t} \quad \big(\textit{i.e.}\ \ \partial_{t,\mathcal{D}}(u)^n_K = \frac{u^n_K - u^{n-1}_K}{\delta t},\ \forall K \in \mathcal{M}\big). \tag{4.10}$$

4.3 A compactness result

The aim of this section is to establish a compactness result for sequences of functions of $\mathrm{H}_{\mathcal{D}}$ which are controlled in the discrete $\mathrm{L}^1(0,T;\mathrm{W}^{1,q}_0(\Omega))$ norm, and the discrete time derivative of which is controlled in the discrete $\mathrm{L}^1(0,T;\mathrm{W}^{-1,r}(\Omega))$ norm, r not being necessarily equal to q'. A new difficulty (with respect with previous analyses which can be found in [11, chapter IV] or [13]) lies in the fact that the space norms for the function and its time-derivative are not dual with respect to the L^2 inner product, which lead us to derive a discrete equivalent of the Lions' lemma 4.3.1 below. Then we prove a discrete analogue of the Aubin-Simon compactness lemma, using the Kolmogorov theorem.

4.3.1 A discrete Lions lemma

Let us first recall the Lions' lemma [17], [5, Lemma II.5.15, p.97].

Lemma 4.3.1. *Let B_0, B_1, B_2 be three Banach spaces such that $B_0 \subset B_1 \subset B_2$, with a compact imbedding of B_0 in B_1 and a continuous imbedding of B_1 in B_2. Then for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that :*

$$\forall u \in B_0, \qquad \|u\|_{B_1} \leq \varepsilon \|u\|_{B_0} + C(\varepsilon)\|u\|_{B_2}.$$

Let us state a discrete version of this lemma suitable for our purpose, that is specifying the norms associated to B_0, B_1 and B_2 as the $\|\cdot\|_{1,q,\mathcal{M}}$ norm, the L^q norm and the $\|\cdot\|_{-1,r,\mathcal{M}}$ norm respectively, acting on a sequence of discrete spaces the step of which tends to zero.

Lemma 4.3.2 (Lions lemma – Discrete L^p version)**.** *Let $\mathcal{M}^k$ be a sequence of meshes the step of which tends to zero, and regular in the sense that any $\mathcal{M}^k$, $k \in \mathbb{N}$, satisfies the regularity assumption* (4.8) *with a unique (i.e. independent of k) real number ξ.*
Let $q, r \in [1, +\infty)$. Then, for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ only depending on ε, q, r and ξ such that, for any sequence of discrete functions $(u^k_{\mathcal{M}})_{k\in\mathbb{N}}$ (i.e. such that, $\forall k \in \mathbb{N}$, $u^k \in \mathrm{H}^k_{\mathcal{M}}$, where $\mathrm{H}^k_{\mathcal{M}}$ is the discrete space associated to $\mathcal{M}^k$):

$$\forall k \in \mathbb{N}, \quad \|u^k\|_{\mathrm{L}^q} \leq \varepsilon\, \|u^k\|_{1,q,\mathcal{M}} + C(\varepsilon)\, \|u^k\|_{-1,r,\mathcal{M}}. \tag{4.11}$$

Proof. Let us suppose that this result is wrong. Then there exists $\varepsilon > 0$ and a sequence of discrete functions $(u^k)_{k\in\mathbb{N}}$ such that:

$$\forall k \in \mathbb{N}, \quad \|u^k\|_{\mathrm{L}^q(\Omega)} \geq \varepsilon \, \|u^k\|_{1,q,\mathcal{M}} + k \, \|u^k\|_{-1,r,\mathcal{M}}. \tag{4.12}$$

Let $(v^k)_{k\in\mathbb{N}}$ be given by:

$$\forall k \in \mathbb{N}, \quad v_k = \frac{1}{\|u^k\|_{\mathrm{L}^q(\Omega)}} \, u^k,$$

so that, $\forall k \in \mathbb{N}$, $\|v^k\|_{\mathrm{L}^q(\Omega)} = 1$. By (4.12), we obtain that:

$$\forall k \in \mathbb{N}, \quad \|v^k\|_{\mathrm{L}^q(\Omega)} \geq \varepsilon \, \|v^k\|_{1,q,\mathcal{M}} + k \, \|v^k\|_{-1,r,\mathcal{M}},$$

and hence:

$$\forall k \in \mathbb{N}, \quad \|v^k\|_{1,q,\mathcal{M}} \leq \frac{1}{\varepsilon}.$$

Thus, thanks to Theorem 4.2.3, possibly up to the extraction of a subsequence, $(v^k)_{k\in\mathbb{N}}$ converges in $\mathrm{L}^q(\Omega)$ when $k \to \infty$ to a limit $v \in W_0^{1,q}$. On one side, this limit satisfies $\|v\|_{\mathrm{L}^q(\Omega)} = 1$. On the other side, we have, for $k \in \mathbb{N}$, $\|v^k\|_{-1,r,\mathcal{M}} \leq 1/k$. Let $\varphi \in \mathrm{C}_c^\infty(\Omega)$. For $k \in \mathbb{N}$, we denote by $\pi^k\varphi$ the discrete function of $\mathrm{H}_{\mathcal{M}}^k$ defined by $(\pi^k\varphi)_K = \varphi(\boldsymbol{x}_K)$, $\forall K \in \mathcal{M}^k$. By the definition of the $\|\cdot\|_{1,r',\mathcal{M}}$ norm, we have:

$$\|\pi^k\varphi\|_{1,r',\mathcal{M}} \leq \|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega)} \, \big(d\,|\Omega|\big)^{1-1/r}.$$

We thus get, for $k \in \mathbb{N}$:

$$\Big|\int v^k \pi^k \varphi \,\mathrm{d}\boldsymbol{x}\Big| \leq \|v^k\|_{-1,r,\mathcal{M}} \, \|\pi^k\varphi\|_{1,r',\mathcal{M}} \leq \|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega)} \, \big(d\,|\Omega|\big)^{1/r'} \, \frac{1}{k},$$

and, passing to the limit when $k \to \infty$, since v^k converges to v in $\mathrm{L}^q(\Omega)$ and $\pi^k\varphi$ converges to φ in $\mathrm{L}^\infty(\Omega)$:

$$\int v\varphi \,\mathrm{d}\boldsymbol{x} = 0.$$

Since this latter relation is valid for any $\varphi \in \mathrm{C}_c^\infty(\Omega)$, this is in contradiction with the fact that $\|v\|_{\mathrm{L}^q(\Omega)} = 1$. ☐

Remark 4.3.3. At first glance, Lemma 4.3.2 may seem to be slightly more general that its continuous counterpart, since it does not require an assumption on the values of q and r which would ensure that $\mathrm{L}^q(\Omega)$ is imbedded in $\mathrm{W}^{-1,r}(\Omega)$. We show in appendix 4.a that the situation is in fact the same at the continuous level, *i.e.* that a continuous counterpart of Inequality (4.11) also holds in the continuous case, for any q and r in $[1, +\infty)$.

4.3.2 Estimation of time translates of discrete functions

Let us first introduce some notations. Let a time step δt be given. For $n \in \mathbb{Z}$, we denote by t^n the time $t^n = n\,\delta t$. Let a mesh $\mathcal{M}$ of Ω be given, and let $\bar{\mathrm{H}}_{\mathcal{D}}$ be the space of discrete functions defined over $\mathbb{R}^d \times \mathbb{R}$ by simply supposing:

(i) that the time discretization covers $\mathbb{R}$, such that $\bar{u} \in \bar{\mathrm{H}}_{\mathcal{D}}$ reads $\bar{u} = (\bar{u}^n)_{n\in\mathbb{Z}}$,

(ii) and that, for $n \in \mathbb{Z}$, $\bar{u}^n$ results from the extension by zero to $\mathbb{R}^d$ of a function of $\mathrm{H}_{\mathcal{M}}$, which we denote by u^n.

For a positive real number τ, let $\chi_\tau^n : \mathbb{R} \to \mathbb{R}$ be the function defined by $\chi_\tau^n(t) = 1$ if $t < t^n < t + \tau$ and $\chi_\tau^n(t) = 0$ otherwise. Then, for a.e. $\boldsymbol{x} \in \mathbb{R}^d$ and a.e. $t \in \mathbb{R}$, the difference $\bar{u}(\boldsymbol{x}, t+\tau) - \bar{u}(\boldsymbol{x}, t)$ can be expanded as follows:

$$\bar{u}(\boldsymbol{x}, t+\tau) - \bar{u}(\boldsymbol{x}, t) = \sum_{n\in\mathbb{Z}} \chi_\tau^n(t) \, \big(u^{n+1}(\boldsymbol{x}) - u^n(\boldsymbol{x})\big). \tag{4.13}$$

In addition, the function χ_τ^n is the characteristic function of the interval $(t^n - \tau, t^n)$ and thus:

$$\forall n \in \mathbb{Z}, \quad \int_{\mathbb{R}} \chi_\tau^n(t) \,\mathrm{d}t = \tau. \tag{4.14}$$

Lemma 4.3.4 (Estimate of the time translates of a function of $\bar{\mathrm{H}}_{\mathcal{D}}$)**.** *Let $\bar{u} = (\bar{u}^n)_{0\le n<N}$ be a function of $\bar{\mathrm{H}}_{\mathcal{D}}$, let $q \in [1,+\infty)$ and $r \in [1,+\infty)$, and let us suppose that there exists a positive real number C such that:*

$$\sum_{n\in\mathbb{Z}} \delta t \, \|u^n\|_{1,q,\mathcal{M}} \le C, \qquad \sum_{n\in\mathbb{Z}} \delta t \, \|\partial_{t,\mathcal{D}}(u)^n\|_{-1,r,\mathcal{M}} \le C.$$

Let $\epsilon > 0$. Let δ be given by:

$$\delta = \frac{\epsilon}{4 \, |\Omega|^{1-1/q} \, C}.$$

Let $C(\delta)$ be such that for any $v \in \mathrm{H}_{\mathcal{M}}$, an inequality of the form of lemma 4.3.2 holds:

$$\|v\|_{\mathrm{L}^q(\Omega)} \le \delta \, \|v\|_{1,q,\mathcal{M}} + C(\delta) \, \|v\|_{-1,r,\mathcal{M}}$$

and, finally, let τ_0 be given by:

$$\tau_0 = \frac{\epsilon}{2 \, |\Omega|^{1-1/q} \, C(\delta) \, C}\ .$$

Then:

$$\forall \tau \le \tau_0, \qquad \|\bar{u}(\cdot, t+\tau) - \bar{u}(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})} \le \epsilon.$$

Proof. Let $\alpha > 0$ and $C(\alpha)$ be a positive real number such that, for any $v \in \mathrm{H}_{\mathcal{M}}$, an inequality of the form of lemma 4.3.2 holds:

$$\|v\|_{\mathrm{L}^q(\Omega)} \le \alpha \, \|v\|_{1,q,\mathcal{M}} + C(\alpha) \, \|v\|_{-1,r,\mathcal{M}}.$$

Let τ be a positive real number, and $\bar{u}$ be a function of $\bar{\mathrm{H}}_{\mathcal{D}}$, its restriction to Ω being denoted by u. We have, for $t \in \mathbb{R}$:

$$\begin{aligned}
&\|\bar{u}(\cdot, t+\tau) - \bar{u}(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|u(\cdot, t+\tau) - u(\cdot,t)\|_{\mathrm{L}^1(\Omega)} \\
&\quad \le |\Omega|^{1-1/q} \, \|u(\cdot, t+\tau) - u(\cdot,t)\|_{\mathrm{L}^q(\Omega)} \\
&\quad \le |\Omega|^{1-1/q} \, \Big[\alpha \, \|u(\cdot, t+\tau) - u(\cdot,t)\|_{1,q,\mathcal{M}} + C(\alpha) \, \|u(\cdot, t+\tau) - u(\cdot,t)\|_{-1,r,\mathcal{M}}\Big].
\end{aligned}$$

This inequality yields:

$$\int_{t\in\mathbb{R}} \|\bar{u}(\cdot, t+\tau) - \bar{u}(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \, \mathrm{d}t \le |\Omega|^{1-1/q} \, \Big[\alpha \, T_1 + C(\alpha) \, T_2\Big],$$

with:

$$T_1 = \int_{t\in\mathbb{R}} \|u(\cdot, t+\tau) - u(\cdot,t)\|_{1,q,\mathcal{M}} \, \mathrm{d}t \quad \text{and} \quad T_2 = \int_{t\in\mathbb{R}} \|u(\cdot, t+\tau) - u(\cdot,t)\|_{-1,r,\mathcal{M}} \, \mathrm{d}t.$$

We have, for T_1:

$$T_1 \le \int_{t\in\mathbb{R}} \|u(\cdot, t+\tau)\|_{1,q,\mathcal{M}} \, \mathrm{d}t + \int_{t\in\mathbb{R}} \|u(\cdot, t)\|_{1,q,\mathcal{M}} \, \mathrm{d}t \le 2\, C.$$

By Identity (4.13), we get for T_2:

$$\begin{aligned}
T_2 \ &\le \int_{t\in\mathbb{R}} \| \sum_{n\in\mathbb{Z}} \chi_\tau^n(t) \ \big(u^{n+1} - u^n\big) \|_{-1,r,\mathcal{M}} \, \mathrm{d}t \\
&= \int_{t\in\mathbb{R}} \| \sum_{n\in\mathbb{Z}} \delta t \ \chi_\tau^n(t) \ \partial_{t,\mathcal{D}}(u)^{n+1}\big) \|_{-1,r,\mathcal{M}} \, \mathrm{d}t.
\end{aligned}$$

By the triangle inequality and Relation (4.14), we thus obtain:

$$T_2 \le \int_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \delta t \ \chi_\tau^n(t) \ \|\partial_{t,\mathcal{D}}(u)^{n+1}\|_{-1,r,\mathcal{M}} \, \mathrm{d}t = \tau \ \sum_{n\in\mathbb{Z}} \delta t \ \|\partial_{t,\mathcal{D}}(u)^{n}\|_{-1,r,\mathcal{M}} = \tau \ C.$$

Gathering the estimates of T_1 and T_2 yields:

$$\int_{t\in\mathbb{R}} \|\bar{u}(\cdot, t+\tau) - \bar{u}(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \, \mathrm{d}t \le C \ |\Omega|^{1-1/q} \ \big(2\,\alpha + \tau \, C(\alpha)\big),$$

and it is now easy to verify that the choice of α and τ suggested by the statement of the lemma yields the desired inequality. The case of negative τ follows by remarking that, by a change of variable in the integration over time, $\|\bar{u}(\cdot, t+\tau) - \bar{u}(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})} = \|\bar{u}(\cdot, t) - \bar{u}(\cdot,t+|\tau|)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})}$. □

4.3.3 A discrete Aubin-Simon lemma

We are now in position to prove the following compactness result.

Theorem 4.3.5. *Let $(u^k)_{k\in\mathbb{N}}$ be a sequence of discrete functions, i.e. a sequence of functions such that, for $k \in \mathbb{N}$, u^k is a function of a space $\mathrm{H}^k_{\mathcal{D}}$ associated to a mesh $\mathcal{M}^k$ and a time step δt^k. We suppose that the sequence of meshes $(\mathcal{M}^k)_{k\in\mathbb{N}}$ is regular, in the sense that the family of regularity parameters $(\xi_{\mathcal{M}^k})_{k\in\mathbb{N}}$ satisfies $\xi_{\mathcal{M}^k} \geq \xi > 0$, $\forall k \in \mathbb{N}$, and that both $h_{\mathcal{M}^k}$ and δt^k tends to zero when k tends to $+\infty$. We suppose that there exists three real numbers $C > 0$, $q \geq 1$ and $r \geq 1$ such that:*

$$\forall k \in \mathbb{N}, \quad \sum_{n=1}^{N^k} \delta t^k \; \|(u^k)^n\|_{1,q,\mathcal{M}} \leq C, \qquad \sum_{n=2}^{N^k} \delta t^k \; \|(\partial_{t,\mathcal{D}}(u)^k)^n\|_{-1,r,\mathcal{M}} \leq C.$$

Then, up to the extraction of a subsequence, the sequence $(u^k)_{k\in\mathbb{N}}$ converges in $\mathrm{L}^1(\Omega \times (0,T))$ to a function $u \in \mathrm{L}^1(0,T;\mathrm{W}^{1,q}_0)$.

Proof. First, we remark that, by Lemma 4.2.1, there exists a real number C_1 only depending on the domain Ω, on the parameter ξ characterizing the regularity of the meshes and on q such that, $\forall k \in \mathbb{N}$, $\forall v \in \mathrm{H}^k_{\mathcal{M}}$ (the discrete space associated to $\mathcal{M}^k$), $\|v\|_{\mathrm{L}^1(\Omega)} \leq C_1 \; \|v\|_{1,q,\mathcal{M}}$. Consequently, we get:

$$\forall k \in \mathbb{N}, \quad \|u^k\|_{\mathrm{L}^1(\Omega\times(0,T))} \leq C_1 \; C.$$

Let φ be a continuously differentiable function from $\mathbb{R}$ to $[0,1]$, such that $\varphi = 1$ on $[0,T]$ and φ is equal to zero on $(-\infty,-T) \cup (2T,+\infty)$. For a given $k \in \mathbb{N}$, let us build a sequence $((\hat{u}^k)^n)_{n\in\mathbb{Z}}$ of functions of $\mathrm{H}^k_{\mathcal{M}}$ as follows:

$$\left|\begin{array}{ll} \text{for } n < -N, & (\hat{u}^k)^n = 0, \\ \text{for } -N \leq n < 0, & (\hat{u}^k)^n = \varphi(t^n) \; (u^k)^{-n}, \\ \text{for } 0 \leq n < N, & (\hat{u}^k)^n = \varphi(t^n) \; (u^k)^n = (u^k)^n, \\ \text{for } N \leq n \leq 2N, & (\hat{u}^k)^n = \varphi(t^n) \; (u^k)^{2N-n}, \\ \text{for } N > 2N, & (\hat{u}^k)^n = 0, \end{array}\right. \tag{4.15}$$

where we have defined $(u^k)^0$ (which does not appear in the statement of the theorem) as $(u^k)^0 = (u^k)^1$. Then we denote by $\bar{u}^k$ the function of $\bar{\mathrm{H}}_{\mathcal{D}}$ (so defined over $\mathbb{R}^d \times \mathbb{R}$) obtained from the sequence $((\hat{u}^k)^n)_{n\in\mathbb{Z}}$. The function $\bar{u}^k$ is equal to u^k on $\Omega \times (0,T)$. Since the function $|\varphi|$ is bounded by 1, we easily get:

$$\|\bar{u}^k\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})} \leq 3 \; \|u^k\|_{\mathrm{L}^1(\Omega\times(0,T))} \leq 3\, C_1\, C, \qquad \sum_{n\in\mathbb{Z}} \delta t \; \|(u^k)^n\|_{1,q,\mathcal{M}} \leq 3C.$$

In addition, we have, for $0 \leq n \leq N-1$:

$$\begin{aligned} \partial_{t,\mathcal{D}}(\bar{u}^k)^{-n} &= \frac{\varphi(t^{-n})\,(\bar{u}^k)^n - \varphi(t^{-n-1})(\bar{u}^k)^{n+1}}{\delta t} \\ &= -\varphi(t^{-n-1}) \; \partial_{t,\mathcal{D}}(\bar{u}^k)^{n+1} - (\bar{u}^k)^n \; \frac{\varphi(t^{-n-1}) - \varphi(t^{-n})}{\delta t}. \end{aligned}$$

Thus, for any $1 \leq s \leq r$, denoting by s' either $s' = 1 - 1/s$ if $s > 1$, or $s' = +\infty$ if $s = 1$:

$$\|\partial_{t,\mathcal{D}}(\bar{u}^k)^{-n}\|_{-1,s,\mathcal{M}} = \sup_{v\in\mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega \partial_{t,\mathcal{D}}(\bar{u}^k)^{-n} \, v \, \mathrm{d}\boldsymbol{x} = T_1 + T_2,$$

with:

$$\begin{aligned} T_1 &= \sup_{v\in\mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega \varphi(t^{-n-1}) \; \partial_{t,\mathcal{D}}(u^k)^{n+1} v \, \mathrm{d}\boldsymbol{x}, \\ T_2 &= \sup_{v\in\mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega (u^k)^n \frac{\varphi(t^{-n-1}) - \varphi(t^{-n})}{\delta t} v \, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

We have for T_1:

$$\begin{aligned} T_1 &= \varphi(t^{-n-1}) \sup_{v\in \mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega \partial_{t,\mathcal{D}}(u^k)^{n+1} v \,\mathrm{d}\boldsymbol{x} \\ &\le \sup_{v\in \mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega \partial_{t,\mathcal{D}}(u^k)^{n+1} v \,\mathrm{d}\boldsymbol{x} = \|\partial_{t,\mathcal{D}}(u^k)^{n+1}\|_{-1,s,\mathcal{M}}, \end{aligned}$$

which is controlled by $\|\partial_{t,\mathcal{D}}(u^k)^{n+1}\|_{-1,r,\mathcal{M}}$ thanks to Inequality (4.7). The term T_2 satisfies:

$$T_2 \le \|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})} \sup_{v\in \mathrm{H}_{\mathcal{M}}} \frac{1}{\|v\|_{1,s',\mathcal{M}}} \int_\Omega (u^k)^n v \,\mathrm{d}\boldsymbol{x}.$$

By Lemma 4.2.1, the $\mathrm{L}^{d/(d-1)}(\Omega)$-norm is controlled by the $\|\cdot\|_{1,q,\mathcal{M}}$-norm, and we can choose s small enough so that the $\|\cdot\|_{1,s',\mathcal{M}}$-norm controls the $\mathrm{L}^{\tilde d}(\Omega)$-norm, where $\tilde d$ is defined by $(1/\tilde d) + ((d-1)/d) = 1$. We then get, by Hölder's inequality:

$$T_2 \le C_2 \ \|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})} \ \|(u^k)^n\|_{1,q,\mathcal{M}}.$$

where C_2 only depends on Ω, ξ, s and q. Applying similar arguments for $N \le n \le 2N$ yields:

$$\sum_{n\in\mathbb{Z}} \delta t^k \ \|(\partial_{t,\mathcal{D}}(\bar u^k)^n\|_{-1,s,\mathcal{M}} \le C_3,$$

where C_3 only depends on φ, Ω, ξ, s, q and C.

We may now apply Lemmas 4.3.2 and 4.3.4 to obtain that, for any $\epsilon > 0$, there exists τ_0 only depending on Ω, ξ C and ϵ such that:

$$\forall \tau \text{ such that } |\tau| \le \tau_0,\ \forall k \in \mathbb{N}, \qquad \|\bar u^k(\cdot, t+\tau) - \bar u^k(\cdot,t)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})} \le \epsilon. \tag{4.16}$$

In addition, from Lemma 4.2.2, we have:

$$\forall k\in\mathbb{N},\ \forall n \in \mathbb{Z},\ \forall \boldsymbol{y} \in \mathbb{R}^d, \quad \|(\bar u^k)^n(\cdot+\boldsymbol{y}) - (\bar u^k)^n(\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d} \le \sqrt{d}\ |\boldsymbol{y}|\ \|(\bar u^k)^n\|_{1,1,\mathcal{M}}.$$

Multiplying by δt^k and summing over the time steps yields:

$$\begin{aligned} &\forall k\in\mathbb{N},\ \forall \boldsymbol{y} \in \mathbb{R}^d, \\ &\quad \|(\bar u^k)^n(\cdot+\boldsymbol{y}) - (\bar u^k)^n(\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R})} \le \sqrt{d}\ |\boldsymbol{y}| \sum_{n\in\mathbb{Z}} \delta t^k\ \|(\bar u^k)^n\|_{1,1,\mathcal{M}}. \end{aligned} \tag{4.17}$$

Since, $\forall k \in \mathbb{N}$, $\bar u^k$ vanishes outside $\Omega \times (-T, 2T)$, Hölder's inequality (4.6) yields:

$$\sum_{n\in\mathbb{Z}} \delta t^n\ \|(\bar u^k)^n\|_{1,1,\mathcal{M}} \le (d\,|\Omega|)^{1/q'} \sum_{n\in\mathbb{Z}} \delta t\ \|(\bar u^k)^n\|_{1,q,\mathcal{M}},$$

which shows that space translates also are uniformly controlled. Kolmogorov's Theorem (e.g. [11, Theorem 14.1 p.833]) thus shows that the sequence $(u^k)_{k\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^1(\Omega\times(0,T))$. The regularity of the limit of subsequences is a consequence of the bound on the space translates, namely the fact that the right hand side of (4.17) is linear with respect to $|\boldsymbol{y}|$ (see [12, Section 5.2.2]). □

Remark 4.3.6 (Regularity of the limit)**.** When the functions of the sequence are more regular than in the assumption of Theorem 4.3.5, so may be also the limit. For instance, if we suppose:

$$\forall k \in \mathbb{N}, \quad \sum_{n=1}^{N^k} \delta t^k\ \|(u^k)^n\|_{1,q,\mathcal{M}}^q \le C,$$

then the limit u lies in the space $\mathrm{L}^q(0,T;\mathrm{W}_0^{1,q}(\Omega))$ (see [12, Section 5.2.2]). This result will be used hereafter.

4.4 The scheme

For $\sigma \in \mathcal{E}_{\rm int}$ and $0 \le n \le N$, let $\mathrm{v}_{K,\sigma}^{n+1/2}$ be defined by:

$$\mathrm{v}_{K,\sigma}^{n+1/2} = \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \int_{\sigma=K|L} \boldsymbol{v}(\boldsymbol{x},t)\cdot \boldsymbol{n}_{K,\sigma}\, \mathrm{d}\gamma(\boldsymbol{x})\, \mathrm{d}t \tag{4.18}$$

The backward first-order in time discretization of (4.1) reads:

$$\begin{aligned} &\forall K \in \mathcal{M},\ \text{ for } 0 \le n < N, \\ &\frac{|K|}{\delta t}\,(u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2}\, u_\sigma^{n+1} + \sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma}\,(u_K^{n+1} - u_L^{n+1}) \\ &\qquad + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \frac{|\sigma|}{d_\sigma}\, u_K^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_K f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \end{aligned} \tag{4.19}$$

where the approximation of u on an internal edge is given by the usual upwind choice:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \quad u_\sigma^{n+1} = \left| \begin{array}{ll} u_K^{n+1} & \text{if } \mathrm{v}_{K,\sigma} \ge 0, \\ u_L^{n+1} & \text{otherwise.} \end{array} \right. \tag{4.20}$$

The initial condition for the scheme is obtained by choosing, for the value of u^0 over a cell $K \in \mathcal{M}$, the mean value of u_0 over K:

$$\forall K \in \mathcal{M}, \qquad u_K^0 = \frac{1}{|K|}\int_K u_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{4.21}$$

4.5 Estimates

Let $\theta \in (1,2)$ and let us define the function ϕ, from $\mathbb{R}$ to $\mathbb{R}$ by:

$$\forall y \in \mathbb{R}, \qquad \phi'(y) = \int_0^y \frac{1}{1+|s|^\theta}\,\mathrm{d}s, \qquad \text{and} \quad \phi(y) = \int_0^y \phi'(s)\,\mathrm{d}s \tag{4.22}$$

The function ϕ enjoys the following features:

1. the function ϕ' is positive over $\mathbb{R}^+$, negative over $\mathbb{R}^-$, and increasing over $\mathbb{R}$; the function ϕ is positive and convex over $\mathbb{R}$.
2. the function $|\phi'|$ is bounded over $\mathbb{R}$; precisely speaking, we have:

$$\forall y \in \mathbb{R}, \quad |\phi'(y)| \le \int_0^1 \mathrm{d}s + \int_1^{+\infty} \frac{1}{s^\theta}\,\mathrm{d}s = 1 + \frac{1}{\theta-1} \tag{4.23}$$

3. Relation (4.23) yields:

$$\forall y \in \mathbb{R}, \quad \phi(y) \le \big(1 + \frac{1}{\theta-1}\big)\ |y| \tag{4.24}$$

   In addition, if we denote by C_ϕ the positive real number defined by $C_\phi = \min(\phi(1), \phi'(1))$, we get, by convexity of ϕ:

$$\forall y \text{ such that } |y| \ge 1, \qquad \phi(y) \ge C_\phi\ |y| \tag{4.25}$$

   In addition, since, for $s \in [0,1]$ and $\theta \in (1,2)$, $1 + |s|^\theta \le 2$, we easily get that $C_\phi \ge 1/4$.

This function ϕ is used in the proof of the following stability result.

Lemma 4.5.1. *Let $u \in \mathrm{H}_{\mathcal{D}}$ be the solution to the scheme* (4.19)-(4.21), *and ϕ be the real function defined over $\mathbb{R}$ by* (4.22). *Then the following bound holds for $1 \leq M \leq N$:*

$$\|u^M\|_{\mathrm{L}^1(\Omega)} + \sum_{n=1}^{M} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (u_K^n - u_L^n) \left[\phi'(u_K^n) - \phi'(u_L^n)\right] + \sum_{n=1}^{M} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{ext}},\\ \sigma\in\mathcal{E}(K)}} \frac{|\sigma|}{d_\sigma} u_K^n \, \phi'(u_K^n) \leq C,$$

where C only depends on Ω, f, u_0 and θ. Since, for any $\theta \in (1,2)$, $s\,\phi'(s) \geq 0$ for $s \in \mathbb{R}$ and ϕ' is an increasing function over $\mathbb{R}$, this inequality provides a bound independent of θ for u in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$.

Proof. Let us take $\phi'(u^{n+1})$ as test-function in the scheme, *i.e.* multiply (4.19) by $\phi'(u_K^{n+1})$ and sum over the control volumes. We get $T_c^{n+1} + T_d^{n+1} = T_f^{n+1}$ with:

$$T_c^{n+1} = \sum_{K\in\mathcal{M}} \phi'(u_K^{n+1}) \Big[\frac{|K|}{\delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2}\, u_\sigma^{n+1}\Big],$$

$$T_d^{n+1} = \sum_{K\in\mathcal{M}} \phi'(u_K^{n+1}) \Big[\sum_{\sigma=K|L} \frac{|\sigma|}{d_\sigma} (u_K^{n+1} - u_L^{n+1}) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma} u_K^{n+1}\Big],$$

$$T_f^{n+1} = \frac{1}{\delta t} \sum_{K\in\mathcal{M}} \phi'(u_K^{n+1}) \int_{t^n}^{t^{n+1}} \int_K f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Since the advection field $\boldsymbol{v}$ is divergence-free, by the definition (4.18), we get:

$$\forall K \in \mathcal{M}, \qquad \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2} = 0.$$

Thanks to Proposition 4.b.1 applied with $\rho_K = \rho_K^* = 1,\ \forall K \in \mathcal{M}$, we thus obtain:

$$T_c^{n+1} \geq \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t} \left[\phi(u_K^{n+1}) - \phi(u_K^n)\right].$$

Reordering the summation in T_d^{n+1}, we have:

$$T_d^{n+1} = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (u_K^{n+1} - u_L^{n+1}) \left[\phi'(u_K^{n+1}) - \phi'(u_L^{n+1})\right] + \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{ext}},\\ \sigma\in\mathcal{E}(K)}} \frac{|\sigma|}{d_\sigma} u_K^{n+1}\, \phi'(u_K^{n+1}).$$

Finally, since ϕ is bounded over $\mathbb{R}^+$ by Relation (4.23), we get:

$$T_f^{n+1} \leq \frac{1}{\delta t} \left(1 + \frac{1}{\theta - 1}\right) \int_{t^n}^{t^{n+1}} \int_\Omega f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Multiplying by δt and summing from $n = 0$ to $n = M - 1$, we thus get:

$$\sum_{K\in\mathcal{M}} |K|\ \phi(u_K^M) + \sum_{n=1}^{M} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma}\,(u_K^n - u_L^n)\,\big[\phi'(u_K^n) - \phi'(u_L^n)\big]$$
$$+ \sum_{n=1}^{M} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext},\\ \sigma\in\mathcal{E}(K)}} \frac{|\sigma|}{d_\sigma}\, u_K^n\, \phi'(u_K^n) \le \sum_{K\in\mathcal{M}} |K|\ \phi(u_K^0)$$
$$+ \Big(1 + \frac{1}{\theta - 1}\Big) \int_0^{t^M} \int_\Omega f(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t,$$

which concludes the proof thanks to the definition (4.21) of u^0, Inequality (4.24) and Inequality (4.25). □

The following lemma is a central argument of estimates in the elliptic case. It may be found in [14], and is recalled here, together with its proof, for the sake of completeness.

Lemma 4.5.2. *Let v be a function of $\mathrm{H}_\mathcal{M}$ and ϕ be the real function defined over $\mathbb{R}$ by (4.22). Let $T_d(v)$ be given by:*

$$T_d(v) = \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma}\,(v_K - v_L)\,\big[\phi'(v_K) - \phi'(v_L)\big] + \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext},\\ \sigma\in\mathcal{E}(K)}} \frac{|\sigma|}{d_\sigma}\, v_K\, \phi'(v_K).$$

Then the following bounds holds for $1 \le p < 2$:

$$\|v\|_{1,p,\mathcal{M}}^p \le \big[T_d(v)\big]^{p/2}\ \Big[C_1 + C_2\, \|v\|_{\mathrm{L}^{\theta p/(2-p)}(\Omega)}^{\theta p/2}\Big],$$

where C_1 and C_2 only depends on p and on the regularity of the mesh, i.e. on the parameter $\xi_\mathcal{M}$ defined by (4.8).

Proof. Let us first introduce some notations. For any face $\sigma \in \mathcal{E}$ and any function $v \in \mathrm{H}_\mathcal{M}$, we define:

$$\partial_\sigma v = \left|\begin{array}{ll} \dfrac{v_K - v_L}{d_\sigma} & \text{if } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \\[2mm] \dfrac{v_K}{d_\sigma} & \text{if } \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K), \end{array}\right. \tag{4.26}$$

and:

$$a_\sigma = \left|\begin{array}{ll} \dfrac{\phi'(v_K) - \phi'(v_L)}{v_K - v_L} & \text{if } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \\[2mm] \dfrac{\phi'(v_K)}{v_K} & \text{if } \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K). \end{array}\right.$$

Note that, for $\sigma \in \mathcal{E}$, the quantity a_σ is non-negative. With these notations, we have:

$$\|v\|_{1,p,\mathcal{M}}^p = \sum_{\sigma\in\mathcal{E}} |\sigma|\, d_\sigma\ |\partial_\sigma v|^p \quad \text{and} \quad T_d(v) = \sum_{\sigma\in\mathcal{E}} |\sigma|\, d_\sigma\, a_\sigma\ (\partial_\sigma v)^2.$$

By Hölder's inequality, we get:

$$\|v\|_{1,p,\mathcal{M}}^p \le \Big[\sum_{\sigma\in\mathcal{E}} |\sigma|\, d_\sigma\, a_\sigma\ (\partial_\sigma v)^2\Big]^{p/2}\ \Big[\sum_{\sigma\in\mathcal{E}} |\sigma|\, d_\sigma\, a_\sigma^{-p/(2-p)}\Big]^{(2-p)/2}. \tag{4.27}$$

For $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, there exists $\bar v_\sigma$ lying between v_K and v_L such that $a_\sigma = \phi''(\bar v_\sigma)$. From the expression of ϕ, we thus get:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \quad \frac{1}{a_\sigma} \le 1 + \max\left(|v_K|,\ |v_L|\right)^\theta$$

By a similar argument, we also have:

$$\forall \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K), \quad \frac{1}{a_\sigma} \le 1 + |v_K|^\theta$$

Inequality (4.27) thus yields $\|v\|^p_{1,p,\mathcal{M}} \le T_d(v)^{p/2}\ T_l^{(2-p)/2}$ with:

$$T_l = \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} |\sigma|\, d_\sigma \left(1+\max\left(|v_K|,\ |v_L|\right)^\theta\right)^{p/(2-p)} + \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext},\\ \sigma\in\mathcal{E}(K)}} |\sigma|\, d_\sigma \left(1+|v_K|^\theta\right)^{p/(2-p)}$$

Using the inequality $(a+b)^\alpha \le 2^\alpha\ (a^\alpha + b^\alpha)$ valid for any positive real numbers a, b and α, we get:

$$2^{-p/(2-p)}\ T_l \le \sum_{\sigma\in\mathcal{E}} |\sigma|\, d_\sigma + \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} |\sigma|\, d_\sigma\ \max\left(|v_K|,\ |v_L|\right)^{\theta p/(2-p)} + \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext},\\ \sigma\in\mathcal{E}(K)}} |\sigma|\, d_\sigma\, |v_K|^{\theta p/(2-p)}$$

Remarking that, for any $K \in \mathcal{M}$, the total weight of the term $|v_K|^{\theta p/(2-p)}$ in the last two sums (summing all its occurrences) is at most equal to $\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, d_\sigma$ and that this quantity is bounded by $C\ |K|$, with C only depending on the regularity of the mesh, we get:

$$2^{-p/(2-p)}\ T_l \le d\ |\Omega| + C \sum_{K\in\mathcal{M}} |K|\ v_K^{\theta p/(2-p)} = d\ |\Omega| + C\ \|v\|^{\theta p/(2-p)}_{\mathrm{L}^{\theta p/(2-p)}(\Omega)}$$

and thus, finally:

$$\|v\|^p_{1,p,\mathcal{M}} \le T_d(v)^{p/2} \left[2^{p/(2-p)} d\ |\Omega| + 2^{p/(2-p)} C\ \|v\|^{\theta p/(2-p)}_{\mathrm{L}^{\theta p/(2-p)}(\Omega)}\right]^{(2-p)/2}$$

which easily yields the desired inequality. □

We are now in position to prove the following estimate, by a technique which is reminiscent of the method used in [4] for the continuous case.

Proposition 4.5.3. *Let $u \in \mathrm{H}_\mathcal{D}$ be the solution to the scheme* (4.19)-(4.21). *Then the following bound holds for $1 \le p < (d+2)/(d+1)$:*

$$\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \le C,$$

where C only depends on Ω, T, f, u_0, p and the regularity of the mesh, i.e. on the parameter $\xi_\mathcal{M}$ defined by (4.8).

Proof. In this proof, we denote by C_i a positive real number only depending on Ω, T, f, u_0, p, θ and the the parameter $\xi_\mathcal{M}$ characterizing the regularity of the mesh. Thanks to Lemma 4.5.2, we get, for $1 \le p < 2$:

$$\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \le \sum_{n=1}^{N} \delta t\ \left[T_d(u^n)\right]^{p/2} \left[C_1 + C_2\ \|u^n\|^{\theta p/2}_{\mathrm{L}^{\theta p/(2-p)}(\Omega)}\right].$$

Since $p < 2$, the discrete Hölder's inequality yields:

$$\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \le \left[\sum_{n=1}^{N} \delta t\ T_d(u^n)\right]^{p/2} \left[\sum_{n=1}^{N} \delta t\ \left[C_1 + C_2\ \|u^n\|^{\theta p/2}_{\mathrm{L}^{\theta p/(2-p)}(\Omega)}\right]^{2/(2-p)}\right]^{(2-p)/2}$$

From Lemma 4.5.1, we know that:

$$\sum_{n=1}^{N} \delta t\ T_d(u^n) \leq C_3.$$

Let us now apply the inequality $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$, valid for $a,\ b,\ \alpha \geq 0$, to obtain:

$$\begin{aligned}\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} &\leq C_3^{p/2}\ \Big[\sum_{n=1}^{N} 2^{2/(2-p)}\delta t\ C_1^{2/(2-p)} \\ &+ \sum_{n=1}^{N} 2^{2/(2-p)}\delta t\ \big[C_2\ \|u^n\|^{\theta p/2}_{L^{\theta p/(2-p)}(\Omega)}\big]^{2/(2-p)}\Big]^{(2-p)/2}\end{aligned}$$

This last relation yields the existence of C_4 and C_5 such that:

$$\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \leq C_4 + C_5\ \Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^{\theta p/(2-p)}_{L^{\theta p/(2-p)}(\Omega)}\Big]^{(2-p)/2}$$

The discrete Sobolev inequality $\|v\|^p_{1,p,\mathcal{M}} \geq C_6\ \|v\|^p_{L^{p^*}(\Omega)}$, which holds for any $v \in H_{\mathcal{M}}$ with $p^* = dp/(d-p)$, yields:

$$C_6 \sum_{n=1}^{N} \delta t\ \|u^n\|^p_{L^{p^*}(\Omega)} \leq \sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \leq C_4 + C_5\ \Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^{\theta p/(2-p)}_{L^{\theta p/(2-p)}(\Omega)}\Big]^{(2-p)/2}$$

The idea to conclude the proof is now to modify the right-hand side of this relation to obtain an inequality of the form:

$$C_6 \sum_{n=1}^{N} \delta t\ \|u^n\|^p_{L^{p^*}(\Omega)} \leq \sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}} \leq C_7 + C_8\ \Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{L^{p^*}(\Omega)}\Big]^\beta$$

with an exponent $\beta < 1$, which will yield a control on $\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{L^{p^*}(\Omega)}$ and, consequently, on $\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{1,p,\mathcal{M}}$. To this purpose, we first use an interpolation inequality, to bound $\|u^n\|_{L^{\theta p/(2-p)}(\Omega)}$ as a function of (a power of) $\|u^n\|_{L^{p^*}(\Omega)}$ and $\|u^n\|_{L^1(\Omega)}$, which is uniformly bounded by Lemma 4.5.1; then an Hölder inequality allows to change the exponent (to p) of this latter norm. Let us recall the interpolation inequality of interest, valid for $1 < r < q$:

$$\forall v \in L^q(\Omega) \quad \|v\|_{L^r(\Omega)} \leq \|v\|^\zeta_{L^q(\Omega)}\ \|v\|^{1-\zeta}_{L^1(\Omega)}, \qquad \text{with } \zeta = \frac{1-1/r}{1-1/q}.$$

Thanks to this inequality and the fact that, for $0 \leq n \leq N$, $\|u^n\|_{L^1(\Omega)} \leq C_9$, we thus get, denoting $r = \theta p/(2-p)$:

$$\Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^{\theta p/(2-p)}_{L^{\theta p/(2-p)}(\Omega)}\Big]^{(2-p)/2} \leq C_{10}\ \Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^{\zeta r}_{L^{p^*}(\Omega)}\Big]^{(2-p)/2} \quad \text{with} \quad \zeta = \frac{1-1/r}{1-1/p^*}.$$

This inequality is valid if $p^* > r$, which is equivalent to $p < (2-\theta)d/(d-\theta)$. We may now apply Hölder's inequality to get:

$$\Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^{\theta p/(2-p)}_{L^{\theta p/(2-p)}(\Omega)}\Big]^{(2-p)/2} \leq C_8\ \Big[\sum_{n=1}^{N} \delta t\ \|u^n\|^p_{L^{p^*}(\Omega)}\Big]^{(2-p)r\zeta/2p}$$

provided that $\zeta r < p$, which reads:

$$\frac{1}{p}\ \frac{r-1}{1-1/p^*} < 1.$$

Expliciting the values of p^* and r as a function of θ and p, it may be seen that this inequality is valid for:

$$p < \frac{(2-\theta)d+2}{d+1}.$$

When this inequality is satisfied, since $1 \leq p < 2$, we have $(2-p)/2 < 1$ and $(2-p)r\zeta/2p < 1$, and we are thus able to conclude the proof as announced. For $d = 1$, $d = 2$ or $d = 3$, we have:

$$\frac{(2-\theta)d+2}{d+1} < \frac{(2-\theta)d}{d-\theta}$$

for θ sufficiently close to one, let us say for $\theta \in (1, \theta_0]$. Let $p \in [1, (d+2)/(d+1))$, and $\theta(p)$ be given by $\theta(p) = \min(\theta_0,\ (\theta_1+1)/2)$ where $\theta_1 \in (1,2)$ is defined by:

$$p = \frac{(2-\theta_1)d+2}{d+1}$$

Then all the inequalities of this proof are valid for $\theta = \theta(p)$, which yields the desired bound. □

Proposition 4.5.4. *Let $u \in \mathrm{H}_{\mathcal{D}}$ be the solution to the scheme* (4.19)-(4.21). *Then the following bound holds:*

$$\sum_{n=1}^{N-1} \delta t\ \|\partial_{t,\mathcal{D}}(u)^n\|_{-1,1,\mathcal{M}} \leq C,$$

where C only depends on Ω, T, f, $\boldsymbol{v}$, u_0 and the regularity of the mesh, i.e. on the parameter $\xi_{\mathcal{M}}$ defined by (4.8).

Proof. Using the notation (4.26), the scheme reads:

$$\forall K \in \mathcal{M},\ \text{for } 0 \leq n < N,$$

$$|K|\, \partial_{t,\mathcal{D}}(u)^n_K = -\sum_{\sigma=K|L} \mathrm{v}^{n+1/2}_{K,\sigma}\, u^{n+1}_\sigma - \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}\, \partial_\sigma u^{n+1} + \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_K f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Let $v \in \mathrm{H}_{\mathcal{M}}$. Multiplying each equation of the scheme by v_K and summing over the control volumes, we get:

$$\int_\Omega \partial_{t,\mathcal{D}}(u)^n\ v\,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} |K|\ \partial_{t,\mathcal{D}}(u)^n_K\ v_K = T_1^{n+1} + T_2^{n+1} + T_3^{n+1},$$

with:

$$T_1^{n+1} = -\sum_{K\in\mathcal{M}} v_K \sum_{\sigma=K|L} \mathrm{v}^{n+1/2}_{K,\sigma}\ u^{n+1}_\sigma, \quad T_2^{n+1} = -\sum_{K\in\mathcal{M}} v_K \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}\,\partial_\sigma u^{n+1},$$
$$T_3^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_\Omega f(\boldsymbol{x},t)\ v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Reordering the sums and supposing, without loss of generality, that any face σ is oriented in such a way that $\mathrm{v}^{n+1/2}_{K,\sigma} \leq 0$, we get for T_1^{n+1}:

$$T_1^{n+1} = -\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} \mathrm{v}^{n+1/2}_{K,\sigma} u^{n+1}_K\ (v_K - v_L),$$

where, by assumption on the velocity field, $|\mathrm{v}^{n+1/2}_{K,\sigma}| \leq \|\boldsymbol{v}\|_{\mathrm{L}^\infty(\Omega)}\ |\sigma|$, $\forall \sigma \in \mathcal{E}_{\mathrm{int}}$. By the discrete Cauchy-Schwarz inequality, we thus get:

$$|T_1^{n+1}| \leq C\ \|u^{n+1}\|_{\mathrm{L}^2(\Omega)}\ \|v\|_{1,2,\mathcal{M}}$$

where C only depends on $\boldsymbol{v}$ and the parameter $\xi_{\mathcal{M}}$ governing the regularity of the mesh.
By a similar computation, we get for T_2^{n+1}:

$$|T_2^{n+1}| = \Big|\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}\ \partial_\sigma u^{n+1}\ \partial_\sigma v\Big| \leq \|u^{n+1}\|_{1,1,\mathcal{M}}\ \|v\|_{1,\infty,\mathcal{M}}.$$

Finally, the term T_3^{n+1} satisfies:

$$|T_3^{n+1}| \leq \frac{1}{\delta t} \; \|v\|_{L^\infty(\Omega)} \; \|f\|_{\Omega\times(t^n,t^{n+1})}.$$

Let $p \in (1,(d+2)/(d+1))$ be such that the discrete $W^{1,p}(\Omega)$ norm of u controls its $L^2(\Omega)$ norm (which, by Lemma 4.2.1) is indeed possible. Since the L^∞ norm is controlled by the $\|\cdot\|_{1,\infty,\mathcal{M}}$ norm, we get, for any $v \in H_\mathcal{M}$:

$$\begin{aligned} \delta t \int_\Omega \partial_{t,\mathcal{D}}(u)^n \; v \,\mathrm{d}\boldsymbol{x} \quad &\leq \delta t\big(|T_1^{n+1}| + |T_2^{n+1}| + |T_3^{n+1}|\big) \\ &\leq C \; \Big[\|u^{n+1}\|_{1,p,\mathcal{M}} + \|f\|_{\Omega\times(t^n,t^{n+1})}\Big] \; \|v\|_{1,\infty,\mathcal{M}}, \end{aligned}$$

which, summing over the time steps and using Proposition 4.5.3, concludes the proof. □

4.6 Convergence analysis

In this section, we prove the following result.

Theorem 4.6.1. *Let $(u^k)_{k\in\mathbb{N}}$ be a sequence of discrete solutions, i.e. a sequence of solutions to the scheme* (4.19)-(4.21)*, with a mesh $\mathcal{M}^k$ and a time step δt^k. We suppose that the sequence of meshes $(\mathcal{M}^k)_{k\in\mathbb{N}}$ is regular, in the sense that the family of regularity parameters $(\xi_{\mathcal{M}^k})_{k\in\mathbb{N}}$ satisfies $\xi_{\mathcal{M}^k} \geq \xi > 0$, $\forall k \in \mathbb{N}$, and that both $h_{\mathcal{M}^k}$ and δt^k tends to zero when k tends to $+\infty$.*

Then, up to the extraction of a subsequence, the sequence $(u^k)_{k\in\mathbb{N}}$ converges in $L^1(\Omega\times(0,T))$ to a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$, for any $p \in [1,(d+2)/(d+1))$, which is a weak solution to the continuous problem, in the sense of Definition 4.1.1.

Proof. Thanks to the estimates of Propositions 4.5.3 and 4.5.4, Theorem 4.3.5 applies, and the sequence $(u^k)_{k\in\mathbb{N}}$ is known to converge in $L^1(\Omega\times(0,T))$ to a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$, for any $p \in [1,(d+2)/(d+1))$. We now show that this function is a weak solution to the continuous problem.

Let $\varphi \in C_c^\infty(\Omega\times[0,T))$. For a given discretization $\mathcal{M}^k$ and δt^k, we denote by φ_K^n the quantity:

$$\forall K \in \mathcal{M}^k, \text{ for } 0 \leq n \leq N^k, \quad \varphi_K^n = \varphi(\boldsymbol{x}_K, t^n).$$

Let us now multiply by φ_K^n each equation of the scheme, multiply by δt and sum over the control volumes and the time steps, to obtain:

$$T_{\partial t}^k + T_c^k + T_d^k = T_f^k$$

with, dropping for short the superscripts k and using the notation (4.26):

$$\begin{aligned} T_{\partial t} \quad &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} |K| \; (u_K^{n+1} - u_K^n) \; \varphi_K^n, \\ T_c \quad &= \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \varphi_K^n \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2} \; u_\sigma^{n+1}, \\ T_d \quad &= \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \varphi_K^n \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma} \, \partial_\sigma u^{n+1}, \\ T_f \quad &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \varphi_K^n \int_{t^n}^{t^{n+1}} \int_K f(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t. \end{aligned}$$

For the first term, we get, reordering the sums:

$$T_{\partial t} = -\sum_{n=1}^{N} \sum_{K\in\mathcal{M}} |K| \; u_K^n \; (\varphi_K^n - \varphi_K^{n-1}) - \sum_{K\in\mathcal{M}} |K| \; u_K^0 \varphi_K^0,$$

which yields:

$$T_{\partial t} = -\int_{t=O}^{T} \int_{\Omega} u(\boldsymbol{x},t)\ \partial_t \varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_1 - \int_{\Omega} u_0(\boldsymbol{x})\ \varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} + R_2,$$

the terms R_1 and R_2 being defined below. The first one reads:

$$R_1 = -\sum_{n=1}^{N} \delta t \sum_{K\in\mathcal{M}} |K|\ u_K^n \Big[\frac{\varphi_K^n - \varphi_K^{n-1}}{\delta t} - \frac{1}{|K|\ \delta t}\int_{t^{n-1}}^{t^n}\int_K \varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\Big]$$

so $|R_1| \leq c_\varphi\ (h+\delta t)\ \|u\|_{\mathrm{L}^1(\Omega\times(0,T))}$ where $c_\varphi = \|\varphi\|_{\mathrm{W}^{1,\infty}(\Omega\times[0,T))}$. The term R_2 reads:

$$R_2 = -\sum_{K\in\mathcal{M}} \int_K u_0(\boldsymbol{x})\ \big[\varphi_K^0 - \varphi(\boldsymbol{x},0)\big]\,\mathrm{d}\boldsymbol{x},$$

so $|R_2| \leq c_\varphi\ h\ \|u_0\|_{\mathrm{L}^1(\Omega)}$.

We now turn to the convection term, which we write $T_c = T_{c,1} + T_{c,2}$, with:

$$T_{c,1} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \varphi_K^n\ u_K^{n+1} \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2},$$

$$T_{c,2} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \varphi_K^n \sum_{\sigma=K|L} \mathrm{v}_{K,\sigma}^{n+1/2}\,(u_\sigma^{n+1} - u_K^{n+1}).$$

By the definition of $\mathrm{v}_{K,\sigma}^{n+1/2}$, the term $T_{c,1}$ reads:

$$\begin{aligned} T_{c,1} &= -\sum_{n=0}^{N-1} \varphi_K^n \int_{t^n}^{t^{n+1}} \int_K u(\boldsymbol{x},t)\ \boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega u(\boldsymbol{x},t)\ \varphi(\boldsymbol{x},t)\ \boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_3, \end{aligned}$$

with:

$$R_3 = -\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_K u(\boldsymbol{x},t)\ \boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t)\ \big[\varphi_K^n - \varphi(\boldsymbol{x},t)\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

so $|R_3| \leq c_\varphi\ h\ \|\boldsymbol{v}\|_{\mathrm{W}^{1,\infty}(\Omega\times(0,T))} \|u\|_{\mathrm{L}^1(\Omega\times(0,T))}$.

We now decompose $T_{c,2} = T_{c,3} + R_4$ where $T_{c,3}$ is chosen to be:

$$\begin{aligned} T_{c,3} &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} u_K \sum_{\sigma=K|L} \int_{t^n}^{t^{n+1}} \int_\sigma \varphi(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}_K\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega u(\boldsymbol{x},t)\ \boldsymbol{\nabla}\cdot\big[\varphi(\boldsymbol{x},t)\ \boldsymbol{v}(\boldsymbol{x},t)\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \end{aligned}$$

and, by difference and reordering the sums, we obtain for R_4:

$$R_4 = -\sum_{n=0}^{N-1} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} (u_K^{n+1} - u_L^{n+1}) \int_{t^n}^{t^{n+1}} \int_\sigma \big[\varphi(\boldsymbol{x},t) - \varphi_L^n\big]\ \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}t$$

where, without loss of generality, we have chosen for the faces the orientation such that $\mathrm{v}_{K,\sigma}^{n+1/2} \geq 0$. We thus get:

$$\begin{aligned} |R_4| &\leq c_\varphi\ h\ \|\boldsymbol{v}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} |\sigma|\ d_\sigma\ \frac{|u_K^{n+1} - u_L^{n+1}|}{d_\sigma} \\ &= c_\varphi\ h\ \|\boldsymbol{v}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \sum_{n=1}^{N} \delta t\ \|u^n\|_{1,1,\mathcal{M}} \end{aligned}$$

Let us now turn to the diffusion term. By a standard reordering of the summations, we get:

$$T_d = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} u_K^{n+1} \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}\, \partial_\sigma \varphi^n,$$

which reads:

$$\begin{aligned} T_d &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} u_K^{n+1} \int_{t^n}^{t^{n+1}} \int_K \Delta\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_5 \\ &= \int_0^T \int_\Omega u(\boldsymbol{x},t)\ \Delta\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_5, \end{aligned}$$

with:

$$R_5 = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} u_K^{n+1} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ R_{K,\sigma},$$

$$R_{K,\sigma} = \frac{\partial_\sigma \varphi^n}{d_\sigma} + \frac{1}{|\sigma|\ \delta t} \int_{t^n}^{t^{n+1}} \int_\sigma \nabla\varphi(\boldsymbol{x},t)\cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}t.$$

For any face $\sigma\in\mathcal{E}_{\rm int}$, $\sigma = K|L$, we have $R_{K,\sigma} = -R_{L,\sigma}$; in addition, for any $K\in\mathcal{M}$ and any $\sigma\in\mathcal{E}(K)$, $|R_{K,\sigma}| \le c_\varphi\ (h+\delta t)$. Hence, reordering once again the sums:

$$R_5 = \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ \partial_\sigma u_{n+1}\ R_{K,\sigma},$$

and:

$$|R_5| \le c_\varphi\ (h+\delta t) \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\ d_\sigma\ \frac{|\partial_\sigma u_{n+1}|}{d_\sigma} = c_\varphi\ (h+\delta t)\ \|u_{n+1}\|_{1,1,\mathcal{M}}.$$

Finally, we have for the last term:

$$T_f = \int_0^T \int_\Omega f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_6,$$

with $|R_6| \le c_\varphi\ (h+\delta t)$.

Finally, gathering all the terms and remarking that $-\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{v}) + \varphi\boldsymbol{\nabla}\cdot(\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\varphi$, we get:

$$\begin{aligned} -\int_{t=O}^T \int_\Omega u(\boldsymbol{x},t)\ \partial_t\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega u_0(\boldsymbol{x})\ \varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} \\ -\int_0^T \int_\Omega u(\boldsymbol{x},t)\ \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \int_0^T \int_\Omega u(\boldsymbol{x},t)\ \Delta\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_0^T \int_\Omega f(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R, \end{aligned}$$

with $|R| \le C\ (h+\delta t)$ where C is controlled by the estimates satisfied by the solution, the regularity of $\boldsymbol{v}$ and φ and independently of the mesh. Letting h and δt tend to zero in this equation thus concludes the proof. □

4.a A version of the Lions and Aubin-Simon lemma

4.a.1 Lions lemma

Lemma 4.a.1. *Let X, B and Y be three Banach spaces satisfying the following hypotheses:*
(i) X is compactly imbedded into B.
(ii) There exists a vectorial space F such that B and Y are imbedded into F and, for all sequence $(u_n)_{n\in\mathbb{N}}$ of $B \cap Y$, if $u_n \to u$ in B and $u_n \to v$ in Y, then $u = v$.
Then, for all $\epsilon > 0$ there exists $C_\epsilon \in \mathbb{R}$ such that $\|u\|_B \leq \epsilon \, \|u\|_X + C_\epsilon \, \|u\|_Y$, for all $u \in X \cap Y$.

Proof. We perform the proof by contradiction. We assume that there exists $\epsilon > 0$ and a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in X \cap Y$ and $1 = \|u_n\|_B > \epsilon\|u_n\|_X + n\|u_n\|_Y$, for all $n \in \mathbb{N}$. Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in X and therefore relatively compact in B. Thus, we can assume that $u_n \to u$ in B and $\|u\|_B = 1$. Furthermore $u_n \to 0$ in Y (since $\|u_n\|_Y \leq 1/n$). Then the second hypothesis of Lemma 4.a.1 gives $u = 0$, which is in contradiction with $\|u\|_B = 1$. ☐

Remark 4.a.2. In some practical case, the second hypothesis of Lemma 4.a.1 may be replaced by the following assumption:
(ii)′ There exists a topological vectorial space F such that B and Y are continuously imbedded into F.
Then:

1. Assumption (ii)′ is stronger than Assumption (ii),
2. Assumption (ii)′ implies the existence of a Banach space G such that B and Y are continuously imbedded into G,
3. as a consequence, Lemma 4.a.1 may be proven using directly a classical lemma due to J.L. Lions.

We first prove the first assertion. Then, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $B \cap Y$ such that $u_n \to u$ in B and $u_n \to v$ in Y. Thanks to the continuous imbedding from B and Y in F, one has $u_n \to u$ in F and $u_n \to v$ in F. Then, $u = v$, which is Assumption (ii).

We now turn to the second point. We set $G = B + Y$ and for $u \in G$, one sets $\|u\|_G = \inf\{\|u_1\|_B + \|u_2\|_Y,\ u = u_1 + u_2,\ u_1 \in B,\ u_2 \in Y\}$. The only difficulty for proving that $\|\cdot\|_G$ is a norm on G is to prove that $\|u\|_G = 0 \Rightarrow u = 0$. Let u be such an element of G, *i.e.* be such that $\|u\|_G = 0$. There exists a sequence $(u_{1,n})_{n\in\mathbb{N}}$ in B and a sequence $(u_{2,n})_{n\in\mathbb{N}}$ in Y such that $u = u_{1,n} + u_{2,n}$, for all $n \in \mathbb{N}$ and $u_{1,n} \to 0$ in B, $u_{2,n} \to 0$ in Y, as $n \to +\infty$. Thus both sequences tend to zero in F, which proves that $u = 0$.
Since both B and Y are continuously imbedded in G (since $\|u\|_B \leq \|u\|_G$ and $\|u\|_Y \leq \|u\|_G$), which is a Banach space, the proof is complete.

Let us now address the third issue. Lemma 4.a.1 is the Lions lemma if $F = Y$. Otherwise, this latter lemma may be applied with G instead of Y and gives that, for all $\epsilon > 0$, there exists $C_\epsilon \in \mathbb{R}$ such that $\|u\|_B \leq \epsilon\|u\|_X + C_\epsilon\|u\|_G$, for all $u \in B$. Since $\|\cdot\|_G \leq \|\cdot\|_Y$, we obtain Lemma 4.a.1.

Remark 4.a.3. We give now an example where the hypotheses of Lemma 4.a.1 are satisfied. Let Ω is a bounded open set of $\mathbb{R}^d$ ($d \geq 1$) with a Lipschitz continuous boundary. Let $1 \leq p, r \leq \infty$, $X = \mathrm{W}^{1,p}(\Omega)$, $B = \mathrm{L}^p(\Omega)$ and $Y = \mathrm{W}^{-1,r}(\Omega) = (\mathrm{W}_0^{1,r'})(\Omega)^\star$, with $r' = r/(r-1)$, that is the (topological) dual space of $W_0^{1,r'}(\Omega)$. The first hypothesis of Lemma 4.a.1 is satisfied. For the second hypothesis, we distinguish the cases $r = 1$ and $r > 1$. In the case $r > 1$, it is possible to take $F = (\mathrm{C}_c^\infty(\Omega))'$, that is the set of linear applications from $\mathrm{C}_c^\infty(\Omega)$ to $\mathbb{R}$ (without any continuity requirement). In the case $r = 1$, the choice $F = (\mathrm{C}_c^\infty(\Omega))'$ is not convenient since $\mathrm{C}_c^\infty(\Omega)$ is not dense in $\mathrm{W}_0^{1,\infty}(\Omega)$ (and therefore two different elements of $W^{-1,1}(\Omega)$ can have the same restriction on $\mathrm{C}_c^\infty(\Omega)$). But, in order to apply Lemma 4.a.1, it is possible to take $F = Y$ since in this case B is imbedded in Y (as usual, one identifies here $u \in B$ with the linear form $\varphi \mapsto \int_\Omega u\varphi \,\mathrm{d}\boldsymbol{x}$, with $\varphi \in \mathrm{C}_c^\infty(\Omega)$ for the case $r > 1$ and $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega)$ in the case $r = 1$).

4.a.2 Aubin-Simon's compactness result

Lemma 4.a.4. *Let X, B and Y be three Banach spaces satisfying the following hypotheses:*

1. X is compactly imbedded into B.

2. There exists a vectorial space F such that B and Y are imbedded into F and, for all sequence $(u_n)_{n\in\mathbb{N}}$ of $B\cap Y$, if $u_n \to u$ in B and $u_n \to v$ in Y, then $u = v$.

Let $T > 0$, $p \in [1,\infty]$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $\mathrm{L}^p((0,T),X)$. Let $q \in [1,\infty]$ and assume that the sequence $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^q((0,T),Y)$. Then, there exists $u \in L^r((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^r((0,T),B)$, as $n \to +\infty$, with $r = \min\{p,q\}$.

Proof. For short, we restrict the exposition to the case where $(ii)'$ holds, which allows a simple proof with the classical Aubin-Simon's compactness result. We take $G = B+Y$ with the norm defined in Remark 4.a.2. we obtain that the sequence $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),G)$ and B is continuously imbedded in G. Then the Aubin-Simon's compactness lemma gives the desired result. □

Remark 4.a.5. We give here some precision on the sense of "$\partial_t u \in \mathrm{L}^q((0,T),Y)$". Let X and Y two Banach spaces and $p,q \in [1,\infty]$. Assuming that $u \in \mathrm{L}^p((0,T),X)$, the weak derivative of u is defined by its action on test functions, that is its action on φ for all $\varphi \in \mathrm{C}_c^\infty((0,T))$ (note that φ takes its values in $\mathbb{R}$). Actually, if $\varphi \in \mathrm{C}_c^\infty((0,T))$, the function $(\partial_t\varphi)\,u$ belongs to $\mathrm{L}^p((0,T),X)$ and therefore to $\mathrm{L}^1((0,T),X)$ and the action of $\partial_t u$ on φ is defined as:

$$\langle \partial_t u, \varphi\rangle = -\int_0^T \partial_t\varphi(t)\ u(t)\,\mathrm{d}t.$$

Note that $\langle \partial_t u, \varphi\rangle \in X$.

In order to give a sense to "$\partial_t u \in \mathrm{L}^q((0,T),Y)$", we assume (as in Lemma 4.a.1 and Lemma 4.a.4) that X and Y are imbedded in the same vectorial space F. Then $\partial_t u \in \mathrm{L}^q((0,T),Y)$ means that there exists $v \in \mathrm{L}^q((0,T),Y)$ (and then v is unique) such that:

$$-\int_0^T \partial_t\varphi(t)\ u(t)\,\mathrm{d}t = \int_0^T \varphi(t)\ v(t)\,\mathrm{d}t \text{ for all } \varphi \in \mathrm{C}_c^\infty((0,T)),$$

this equality making sense since:

$$\int_0^T \partial_t\varphi(t)\ u(t)\,\mathrm{d}t \in X \subset F \quad \text{and} \quad \int_0^T \varphi(t)\ v(t)\,\mathrm{d}t \in Y \subset F.$$

4.b A stability result for a general class of convection operators

In a compressible flow, the mass balance reads:

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0. \tag{4.28}$$

Let ϕ be a regular real function, and let us suppose that ρ, z and $\boldsymbol{v}$ are regular scalar (for ρ and z) and vector-valued ($\boldsymbol{v}$) fields, and that $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Then we have:

$$\begin{aligned}\int_\Omega \phi'(z)\big[\partial_t(\rho z) + \boldsymbol{\nabla}\cdot(\rho z\boldsymbol{v})\big]\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \rho\phi'(z)\,\partial_t z + \rho\phi'(z)\nabla z\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ = \int_\Omega \rho\,\partial_t\big[\phi(z)\big] + \rho\,\nabla\big[\phi(z)\big]\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \rho\,\partial_t\phi(z) - \phi(z)\,\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\\ = \int_\Omega \rho\,\partial_t\phi(z) + \phi(z)\,\partial_t\rho\,\mathrm{d}\boldsymbol{x} &= \frac{d}{dt}\int_\Omega \rho\,\phi(z)\,\mathrm{d}\boldsymbol{x}\end{aligned} \tag{4.29}$$

Taking for z one component of the velocity itself and $\phi(s) = s^2/2$, this computation is the central argument of the so-called kinetic energy conservation theorem. If z satisfies $\partial_t(\rho z) + \boldsymbol{\nabla}\cdot(\rho z\boldsymbol{v}) - \boldsymbol{\nabla}\cdot(\lambda\nabla z) = 0,\ \lambda \geq 0$, choosing $\phi(s) = \min(0,s)^2$ yields the fact that z remains non-negative, if its initial condition is non-negative (which can also be seen by noting that, thanks to (4.28), we have $\partial_t(\rho z) + \boldsymbol{\nabla}\cdot(\rho z\boldsymbol{v}) = \rho\big[\partial_t z + \boldsymbol{v}\cdot\nabla z\big]$, and this latter operator is known to satisfy a maximum principle).

The aim of this section is to prove a discrete analogue to (4.29). We thus generalize the proofs already given for the specific choices for ϕ mentionned above, namely for $\phi(s) = s^2/2$ in [15] and for $\phi(s) = \min(0,s)^2$ in [16].

Proposition 4.b.1. *Let* $(\rho_K)_{K\in\mathcal{M}}$, $(\rho^*_K)_{K\in\mathcal{M}}$, $(F_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm int}}$ *three families of real numbers such that:*

$$\begin{aligned} &(i) \quad \forall K \in \mathcal{M},\ \rho_K \geq 0,\ \rho^*_K \geq 0,\\ &(ii) \quad \forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L,\ F_{K,\sigma} = -F_{L,\sigma},\\ &(iii) \quad \forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t}\ (\rho_K - \rho^*_K) + \sum_{\sigma=K|L} F_{K,\sigma} = 0. \end{aligned} \tag{4.30}$$

Let ϕ *be a real convex function defined over* $\mathbb{R}$*, and let* $(z_K)_{K\in\mathcal{M}}$ *and* $(z^*_K)_{K\in\mathcal{M}}$ *be two families of real numbers. For* $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$*, we denote by* z_σ *the quantity defined by* $z_\sigma = z_K$ *if* $F_{K,\sigma} \geq 0$ *and* $z_\sigma = z_L$ *otherwise. Then:*

$$\sum_{K\in\mathcal{M}} \phi'(z_K)\ \Big[\frac{|K|}{\delta t}\ (\rho_K\, z_K - \rho^*_K\, z^*_K) + \sum_{\sigma=K|L} F_{K,\sigma}\, z_\sigma\Big] \geq \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \Big[\rho_K\, \phi(z_K) - \rho^*_K\, \phi(z^*_K)\Big].$$

Proof. Let us write:

$$\sum_{K\in\mathcal{M}} \phi'(z_K)\ \Big[\frac{|K|}{\delta t}\ (\rho_K\, z_K - \rho^*_K\, z^*_K) + \sum_{\sigma=K|L} F_{K,\sigma}\, z_\sigma\Big] = T_1 + T_2$$

with:

$$T_1 = \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \phi'(z_K)\ (\rho_K\, z_K - \rho^*_K\, z^*_K), \qquad T_2 = \sum_{K\in\mathcal{M}} \phi'(z_K)\Big[\sum_{\sigma=K|L} F_{K,\sigma}\, z_\sigma\Big].$$

The first term may be split as $T_1 = T_{1,1} + T_{1,2}$ with:

$$T_{1,1} = \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \phi'(z_K)\, z_K\, (\rho_K - \rho^*_K), \qquad T_{1,2} = \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \rho^*_K\, \phi'(z_K)\, (z_K - z^*_K).$$

By convexity of ϕ, we get:

$$T_{1,2} \geq \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \rho^*_K\, (\phi(z_K) - \phi(z^*_K)).$$

The second term reads $T_2 = T_{2,1} + T_{2,2} + T_{2,3}$ with:

$$T_{2,1} = \sum_{K\in\mathcal{M}} \phi'(z_K)\, z_K\Big[\sum_{\sigma=K|L} F_{K,\sigma}\Big], \qquad T_{2,2} = \sum_{K\in\mathcal{M}} \phi(z_K)\Big[-\sum_{\sigma=K|L} F_{K,\sigma}\Big],$$
$$T_{2,3} = \sum_{K\in\mathcal{M}}\Big[\sum_{\sigma=K|L} F_{K,\sigma}\big(\phi(z_K) + \phi'(z_K)\,(z_\sigma - z_K)\big)\Big].$$

By Relation (iii) of (4.30), the terms $T_{1,1}$ and $T_{2,1}$ cancel, and $T_{2,2}$ may be written as:

$$T_{2,2} = \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ \phi(z_K)\, (\rho_K - \rho^*_K).$$

Reordering the sums in $T_{2,3}$, we get:

$$T_{3,3} = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} F_{K,\sigma}\big[\phi(z_K) + \phi'(z_K)\,(z_\sigma - z_K) - \phi(z_L) - \phi'(z_L)\,(z_\sigma - z_L)\big].$$

Without loss of generality, let us suppose that we have chosen, in the last sum, the orientation of $\sigma = K|L$ in such a way that $F_{K,\sigma} \geq 0$. We thus get, since $z_\sigma = z_K$:

$$\begin{aligned} \delta\phi|_\sigma &= \phi(z_K) + \phi'(z_K)\,(z_\sigma - z_K) - \phi(z_L) - \phi'(z_L)\,(z_\sigma - z_L)\\ &= \phi(z_K) - \big[\phi(z_L) + \phi'(z_L)\,(z_K - z_L)\big], \end{aligned}$$

which is non-negative by convexity of ϕ. Finally, we thus get:

$$T_1 + T_2 \geq \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t} \left[\rho_K^* \left(\phi(z_K) - \phi(z_K^*)\right) + \phi(z_K)\left(\rho_K - \rho_K^*\right)\right],$$

which concludes the proof. □

Remark 4.b.2 (Other choices for z_σ)**.** In fact, Inequality of Proposition 4.b.1 holds for any choice such that the quantity $\delta\phi|_\sigma$ defined in its proof is non-negative, which may be written as:

$$\left[\phi'(z_K) - \phi'(z_L)\right]\,(z_\sigma - z_K) \geq \phi(z_L) + \phi'(z_L)\,(z_K - z_L) - \phi(z_K).$$

Let us suppose that ϕ is twice continuously differentiable. There exists $\bar{z}_\sigma$ and $\bar{\bar{z}}_\sigma$, both lying between z_K and z_L and such that:

$$\begin{aligned} &\phi'(z_K) = \phi'(z_L) + \phi''(\bar{z}_\sigma)\,(z_K - z_L)^2, \\ &\phi(z_K) = \phi(z_L) + \phi'(z_L)\,(z_K - z_L) + \tfrac{1}{2}\phi''(\bar{\bar{z}}_\sigma)\,(z_K - z_L)^2. \end{aligned}$$

Let us now define θ such that $z_\sigma - z_K = \theta\,(z_K - z_L)$. With this notations, we obtain that $\delta\phi|_\sigma \geq 0$ is equivalent to:

$$\theta \leq \frac{1}{2}\,\frac{\phi''(\bar{\bar{z}}_\sigma)}{\phi''(\bar{z}_\sigma)}.$$

By convexity of ϕ, the upwind choice (*i.e.* $\theta = 0$) always satisfies this relation, which is consistent with Proposition 4.b.1. In addition, we see that, for $\phi(s) = s^2$, the choice $\theta = 1/2$ is possible: in other words, as proven in [15], the centered choice ensures the kinetic energy conservation.

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Appendices

Annexe A

Analysis of a fractional-step scheme for the P_1 radiative diffusion model

ANALYSIS OF A FRACTIONAL-STEP SCHEME FOR THE $\mathbf{P}_1$ RADIATIVE DIFFUSION MODEL

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Abstract.We address in this paper a nonlinear parabolic system, which is built to retain the main mathematical difficulties of the P1 radiative diffusion physical model. We propose a finite volume fractional-step scheme for this problem enjoying the following properties. First, we show that each discrete solution satisfies *a priori* L^∞-estimates, through a discrete maximum principle; by a topological degree argument, this yields the existence of a solution, which is proven to be unique. Second, we establish uniform (with respect to the size of the meshes and the time step) L^2-bounds for the space and time translates; this proves, by the Kolmogorov theorem, the relative compactness of any sequence of solutions obtained through a sequence of discretizations the time and space steps of which tend to zero; the limits of converging subsequences are then shown to be a solution to the continuous problem. Estimates of time translates of the discrete solutions are obtained through the formalization of a generic argument, interesting for its own sake.

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Introduction

We address in this paper the following nonlinear parabolic system:

$$\left|\begin{array}{ll} \dfrac{\partial u}{\partial t} - \Delta u + u^4 - \varphi = 0 & \text{for } (x,t) \in \Omega \times (0,T), \\ \varphi - \Delta\varphi - u^4 = 0 & \text{for } (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T), \\ \boldsymbol{\nabla}\varphi \cdot n = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T), \end{array}\right. \tag{A.1}$$

where Ω is a connected bounded subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, which is supposed to be polygonal ($d = 2$) or polyhedral ($d = 3$), $T < \infty$ is the final time, u and φ are two real-valued functions defined on $\Omega \times [0, T)$ and $\partial\Omega$ stands for the boundary of Ω of outward normal n. The initial value for u, denoted by u_0, is supposed to lie in $\mathrm{L}^\infty(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and to satisfy $u_0(x) \geq 0$ for almost every $x \in \Omega$.

This system of partial differential equations is inspired from a simplified radiative transfer physical model, the so-called $\mathbf{P}_1$ model, sometimes used in computational fluid dynamics for the simulation of high temperature optically thick flows, as encountered for instance in fire modelling (see e.g. [11] for an exposition of the theory, [10, 2, 9, 12] for recent developments and applications or the documentation of the CFX or FLUENT commercial codes for a synthetic description). In this context, the unknown u stands for the temperature, φ for the radiative intensity and the first equation is the energy balance. System (A.1) has been derived with the aim of retaining the main mathematical difficulties of the initial physical model; in particular, adding a convection term in the first equation would only require minor changes in the theory developed hereafter.

In this short paper, we give a finite volume scheme for the discretization of (A.1) and prove the existence and uniqueness of the discrete solution and its convergence to a solution of (A.1), thus showing that this problem indeed admits a solution, in a weak sense which will be defined.

A.1 The finite volume scheme

Even though the arguments developed in this paper are valid for any general admissible discretization in the sense of Definition 9.1, p. 762 in [4], we choose, for the sake of simplicity, to restrict the presentation to simplicial meshes. We thus suppose given a triangulation $\mathcal{M}$ of Ω, that is a finite collection of d-simplicial control volumes K, pairwise disjoint, and such that $\bar{\Omega} = \cup_{K\in\mathcal{M}}\bar{K}$; the mesh is supposed to be conforming in the sense that two neighbouring simplices share a whole face (*i.e.* there is no hanging node). In addition, we assume that, for any $K \in \mathcal{M}$, the circumcenter x_K of K lies in K; note that, for each neighbouring control volumes K and L, the segment $[x_K, x_L]$ is orthogonal to the face $K|L$ separating K from L.

For each simplex K, we denote by $\mathcal{E}(K)$ the set of the faces of K and by $|K|$ the measure of K. The set of faces of the mesh $\mathcal{E}$ is split into the set $\mathcal{E}_{\rm int}$ of internal ones (*i.e.* separating two control volumes) and the set $\mathcal{E}_{\rm ext}$ of faces included in the domain boundary. For each internal face $\sigma = K|L$, we denote by $|\sigma|$ the (d-1)-dimensional measure of σ and by d_σ the distance $d(x_K, x_L)$; for an external face σ of a control volume K, d_σ stands for distance from x_K to σ. The regularity of the mesh is characterized by the parameter $\theta_\mathcal{M}$ defined by:

$$\theta_\mathcal{M} \stackrel{\rm def}{=} \min_{K\in\mathcal{M}} \frac{\rho_K}{h_K} \tag{A.2}$$

where ρ_K and h_K stands for the diameter of the largest ball included in K and the diameter of K, respectively.

We denote by $\mathrm{H}_\mathcal{M}(\Omega) \subset \mathrm{L}^2(\Omega)$ the space of functions which are piecewise constant over each control volume $K \in \mathcal{M}$. For all $u \in \mathrm{H}_\mathcal{M}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by u_K the constant value of u in K. The space $\mathrm{H}_\mathcal{M}(\Omega)$ is equipped with the following Euclidean structure. For $(u,v) \in (\mathrm{H}_\mathcal{M}(\Omega))^2$, we define the following inner product:

$$[u,v]_\mathcal{M} \stackrel{\rm def}{=} \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{|\sigma|}{d_\sigma}\,(u_L-u_K)(v_L-v_K) + \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))} \frac{|\sigma|}{d_\sigma}\,u_K\,v_K. \tag{A.3}$$

Thanks to the discrete Poincaré inequality (A.5) given below, this scalar product defines a norm on $\mathrm{H}_\mathcal{M}(\Omega)$:

$$\|u\|_{1,\mathcal{M}} \stackrel{\rm def}{=} [u,u]_\mathcal{M}^{1/2}. \tag{A.4}$$

The following discrete Poincaré inequality holds (see lemma 9.1, p. 765, in [4]):

$$\|u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\ \|u\|_{1,\mathcal{M}} \qquad \forall u \in \mathrm{H}_\mathcal{M}(\Omega). \tag{A.5}$$

We also define the following semi-inner product and semi-norm:

$$< u,v >_\mathcal{M} \stackrel{\rm def}{=} \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{|\sigma|}{d_\sigma}\,(u_L-u_K)(v_L-v_K), \qquad |u|_{1,\mathcal{M}} \stackrel{\rm def}{=} < u,u >_\mathcal{M}^{1/2}.$$

These inner products can be seen as discrete analogues to the standard H^1-inner product, with, in the first one, an implicitly assumed zero boundary condition.
For any function $u \in \mathrm{H}_\mathcal{M}(\Omega)$, we also define the following discrete H^{-1}-norm:

$$\|u\|_{-1,\mathcal{M}} \stackrel{\rm def}{=} \sup_{v\in\mathrm{H}_\mathcal{M}(\Omega),\, v\neq 0} \frac{\displaystyle\int_\Omega u\,v\,\mathrm{d}\boldsymbol{x}}{\|v\|_{1,\mathcal{M}}}.$$

By inequality (A.5), the $\|\cdot\|_{-1,\mathcal{M}}$-norm is controlled by the $\mathrm{L}^2(\Omega)$-norm.

The discrete Laplace operators associated with homogeneous Dirichlet and Neumann boundary conditions, denoted by $\Delta_{\mathcal{M},\mathrm{D}}(\cdot)$ and $\Delta_{\mathcal{M},\mathrm{N}}(\cdot)$ respectively, are defined as follows:

$$\begin{aligned} &\forall \psi \in \mathrm{H}_\mathcal{M}(\Omega),\\ &(\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K = \frac{1}{|K|}\sum_{\sigma=K|L}\frac{|\sigma|}{d_\sigma}\,(\psi_L-\psi_K),\\ &(\Delta_{\mathcal{M},\mathrm{D}}(\psi))_K = (\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}\frac{|\sigma|}{d_\sigma}\,(-\psi_K). \end{aligned} \tag{A.6}$$

The links between these operators and the above defined inner products is clarified by the following identities:

$$\forall \psi \in \mathrm{H}_{\mathcal{M}}(\Omega), \qquad \sum_{K\in\mathcal{M}} -|K|\ \psi_K\ (\Delta_{\mathcal{M},\mathrm{N}}(\psi))_K =< \psi,\psi >_{\mathcal{M}}$$

$$\text{and} \qquad \sum_{K\in\mathcal{M}} -|K|\ \psi_K\ (\Delta_{\mathcal{M},\mathrm{D}}(\psi))_K = [\psi,\psi]_{\mathcal{M}}.$$

Finally, we suppose given a partition of the interval $(0,T)$, which we assume regular for the sake and simplicity, with $t^0 = 0, \ldots, t^n = n\,\delta t, \ldots t^N = T$.

A each time t_n, an approximation of the solution $(u^n, \varphi^n) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{H}_{\mathcal{M}}(\Omega)$ is given by the following finite volume scheme:

$$\forall K \in \mathcal{M},\quad \left|\begin{array}{ll} (i) & \dfrac{u_K^{n+1}-u_K^n}{\delta t} - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K + |u_K^{n+1}|\ (u_K^{n+1})^3 - \varphi_K^n = 0, \\ (ii) & \varphi_K^{n+1} - (\Delta_{\mathcal{M},\mathrm{N}}(\varphi^{n+1}))_K - (u_K^{n+1})^4 = 0. \end{array}\right. \tag{A.7}$$

In the first equation, the term u^4 is discretized as $|u_K^{n+1}|\ (u_K^{n+1})^3$ to ensure positivity (see proof of Proposition A.2.1 and Remark A.2.2 below). The scheme is initialized as follows:

$$\forall K \in \mathcal{M}, \quad u_K^0 = \frac{1}{|K|}\int_K u_0(x)\ \mathrm{d}\boldsymbol{x} \tag{A.8}$$

and φ^0 is given by (A.7)-(ii), where we set $n+1=0$.

A.2 A priori L^∞ estimates, existence and uniqueness of the discrete solution

We prove in this section the following result.

Proposition A.2.1.

1. *The scheme* (A.7) *has a unique solution.*
2. *For* $0 \le n \le N$*, the unknown* φ^n *satisfies the following estimate:*

$$\forall K \in \mathcal{M}, \quad 0 \le \varphi_K^n \le \left[\max_{L\in\mathcal{M}} u_L^n\right]^4.$$

3. *For* $1 \le n \le N$*, the unknown* u^n *satisfies the following estimate:*

$$\forall K \in \mathcal{M}, \quad 0 \le u_K^n \le \max_{L\in\mathcal{M}} u_L^{n-1}.$$

Proof. Step one: positivity of the unknowns.

We first observe that, from its definition (A.8) and thanks to the fact that u_0 is non-negative, u^0 is a non-negative function. Let us suppose that this property still holds at time step n. We write the second equation of the scheme (A.7) as:

$$\varphi_K^n - (\Delta_{\mathcal{M},\mathrm{N}}(\varphi^n))_K = (u_K^n)^4.$$

This set of relations is a linear system for φ^n, the matrix of which is an M-matrix: indeed, from the definition (A.6) of $\Delta_{\mathcal{M},\mathrm{N}}(\cdot)$, it can be easily checked that its diagonal is strictly dominant and has only positive entries, and all its off-diagonal entries are non-positive. Since the right-hand side of this equation is non-negative, φ^n is also non-negative. The first equation of the scheme (A.7) now can be recast as:

$$\left[\frac{1}{\delta t} + |u_K^{n+1}|\ (u_K^{n+1})^2\right] u_K^{n+1} - (\Delta_{\mathcal{M},\mathrm{D}}(u^{n+1}))_K = \frac{1}{\delta t} u_K^n + \varphi_K^n. \tag{A.9}$$

This set of relations can be viewed as a matrix system for the unknown u^{n+1} the matrix of which depends on u^{n+1} but is also an M-matrix whatever u^{n+1} may be. Since we know that $\varphi^n \geq 0$, the right-hand side of this equation is by assumption non-negative and so is u^{n+1} too.

Step two: upper bounds.

Let $\bar\varphi$ be a constant function of $H_{\mathcal{M}}(\Omega)$. From the definition (A.6) of $\Delta_{\mathcal{M},N}(\cdot)$, we see that $\Delta_{\mathcal{M},N}(\bar\varphi) = 0$ and the second relation of the scheme thus implies:

$$(\varphi^{n+1} - \bar\varphi)_K - (\Delta_{\mathcal{M},N}(\varphi^{n+1} - \bar\varphi))_K = (u_K^{n+1})^4 - \bar\varphi_K.$$

Choosing for the constant value of $\bar\varphi$ the quantity $(\max_{K\in\mathcal{M}} u_K^{n+1})^4$ yields a non-positive right-hand side, and so, from the above mentionned property of the matrix of this linear system, $(\varphi^{n+1} - \bar\varphi)_K \leq 0$, $\forall K \in \mathcal{M}$, which equivalently reads:

$$\varphi_K^{n+1} \leq (\max_{L\in\mathcal{M}} u_L^{n+1})^4, \quad \forall K \in \mathcal{M}. \tag{A.10}$$

Let us now turn to the estimate of the first unknown u^{n+1}. Let K_0 be a control volume where u^{n+1} reaches its maximum value. From the definition (A.6) of $\Delta_{\mathcal{M},D}(\cdot)$, it appears that:

$$-(\Delta_{\mathcal{M},D}(u^{n+1}))_{K_0} \geq 0.$$

The first relation of the scheme reads:

$$\frac{1}{\delta t}\,(u_{K_0}^{n+1} - u_{K_0}^{n}) - (\Delta_{\mathcal{M},D}(u^{n+1}))_{K_0} + \left[|u_{K_0}^{n+1}|\,(u_{K_0}^{n+1})^3 - \varphi_{K_0}^n\right] = 0. \tag{A.11}$$

By the inequality (A.10), we see that supposing that $u_{K_0}^{n+1} > \max_{K\in\mathcal{M}} u_K^n$ yields that the first and third term of the preceding relation are positive, while the second one is non-negative, which is in contradiction with the fact that their sum is zero.

Step three: existence of a solution.

Let us suppose that we have obtained a solution to the scheme up to time step n. Let the function $F(\cdot)$ be defined as follows:

$$\left|\begin{array}{l} \mathbb{R}^{\mathrm{card}(\mathcal{M})} \times [0,1] \quad \longrightarrow \quad \mathbb{R}^{\mathrm{card}(\mathcal{M})}, \\ ((u_K)_{K\in\mathcal{M}},\ \alpha) \quad \mapsto \quad (v_K)_{K\in\mathcal{M}} \quad \text{such that:} \\ \quad \forall K \in \mathcal{M}, \quad v_K = \dfrac{1}{\delta t}\,(u_K - u_K^n) - (\Delta_{\mathcal{M},D}(u))_K + \alpha\left[|u_K|\,(u_K)^3 - \varphi_K^n\right]. \end{array}\right.$$

The solution u^{n+1} of the first relation of the scheme is the solution to:

$$F((u_K^{n+1})_{K\in\mathcal{M}}, 1) = 0. \tag{A.12}$$

First, we observe that the function F_0, which maps $\mathbb{R}^{\mathrm{card}(\mathcal{M})}$ onto $\mathbb{R}^{\mathrm{card}(\mathcal{M})}$ and is defined by $F_0((u_K)_{K\in\mathcal{M}}) = F((u_K)_{K\in\mathcal{M}}, 0)$, is affine and one-to-one. Second, we see from their proofs that the estimates on u^{n+1} proven in step one and step two for $\alpha = 1$ in fact holds uniformly for $\alpha \geq 0$. The existence of a solution to (A.12) then follows by a topological degree argument (see e.g. [3]).
Finally, the existence (and uniqueness) of the solution to the second equation of the scheme, which is a linear system, follows from the above mentionned properties of the associated matrix.

Step four: uniqueness of the solution.

Let us suppose that the solution is unique up to step n and that there exist two solutions u^{n+1} and v^{n+1} to the first equation of the scheme. By the identity $a^4 - b^4 = (a-b)\,(a^3 + a^2 b + ab^2 + +b^3)$, the difference $\delta u = u^{n+1} - v^{n+1}$ satisfies the following system of equations:

$$\forall K \in \mathcal{M}, \quad \left[\frac{1}{\delta t} + \left((u_K^{n+1})^3 + (u_K^{n+1})^2 v_K^{n+1} + u_K^{n+1}(v_K^{n+1})^2 + (v_K^{n+1})^3\right)\right]\ \delta u_K \\ - (\Delta_{\mathcal{M},D}(\delta u))_K = 0.$$

Since we know from the precedent analysis that both u_K^{n+1} and v_K^{n+1} are non-negative, this set of relations can be seen as a matrix system for δu the matrix of which is an M-matrix; we thus get $\delta u = 0$, which proves the uniqueness of the solution. □

Remark A.2.2. We see from Relation (A.9) that the discretization of u^4 as a product of a positive quantity (here $|u^{n+1}|^p$) and $(u^{n+1})^q$ (where $p+q=4$) with q odd (*i.e.* $q=3$ or $q=1$) is essential to prove the non-negativity of u^{n+1}; note that we have indeed observed in practice some (non-physical) negative values when this term is discretized as $(u^{n+1})^4$.

A.3 Convergence to a solution of the continuous problem

Let $\mathrm{H}_{\mathcal{D}}$ be the space of piecewise constant functions over each $K\times I^n$, for $K\in\mathcal{M}$ and $I^n=[t^n,\ t^{n+1})$, $0\le n\le N-1$. To each sequence $(u^n)_{n=0,N}$ of functions of $\mathrm{H}_{\mathcal{M}}(\Omega)$, we associate the function $u\in\mathrm{H}_{\mathcal{D}}$ defined by $u(x,t)=u^n(x)$ for $t^n\le t<t^{n+1}$, $0\le n\le N-1$. In addition, for any $u\in\mathrm{H}_{\mathcal{D}}$, we define $\partial_{t,\mathcal{D}}(u)\in\mathrm{H}_{\mathcal{D}}$ by $\partial_{t,\mathcal{D}}(u)(x,t)=\partial_{t,\mathcal{D}}(u)^n(x)$ for $t^n\le t<t^{n+1}$, $0\le n\le N-1$ where the function $\partial_{t,\mathcal{D}}(u)^n\in\mathrm{H}_{\mathcal{M}}(\Omega)$ is defined by:

$$\partial_{t,\mathcal{D}}(u)^n(x)\stackrel{\text{def}}{=}\frac{u^{n+1}(x)-u^n(x)}{\delta t}\qquad\Big(i.e.\ \ \partial_{t,\mathcal{D}}(u)^n_K=\frac{u^{n+1}_K-u^n_K}{\delta t},\quad\forall K\in\mathcal{M}\Big).$$

For any function $u\in\mathrm{H}_{\mathcal{D}}$, we define the following norms and semi-norms:

$$\|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}\stackrel{\text{def}}{=}\delta t\sum_{n=0}^{N}\|u^n\|^2_{1,\mathcal{M}},$$
$$\|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})}\stackrel{\text{def}}{=}\delta t\sum_{n=0}^{N-1}\|u^n\|^2_{-1,\mathcal{M}},$$
$$|u|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}\stackrel{\text{def}}{=}\delta t\sum_{n=0}^{N}|u^n|^2_{1,\mathcal{M}}.$$

The norms $\|\cdot\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and $|\cdot|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ can be seen as discrete equivalents of the $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$-norm, and $\|\cdot\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})}$ may be considered as a discrete $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$-norm.

The following result provides estimates of the solution to the considered scheme.

Proposition A.3.1 (Estimates in energy norms)**.** *Let u and φ be the functions of $\mathrm{H}_{\mathcal{D}}$ associated to $(u^n)_{0\le n\le N}\in\mathrm{H}_{\mathcal{M}}(\Omega)^{N+1}$ and $(\varphi^n)_{0\le n\le N}\in\mathrm{H}_{\mathcal{M}}(\Omega)^{N+1}$ respectively, themselves being given by the scheme* (A.7) *and the initial condition* (A.8)*. Then the following estimate holds:*

$$\begin{aligned}\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}+\|\partial_{t,\mathcal{D}}(u)\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_{\mathcal{M}})}+\|\varphi\|_{L^2((0,T)\times\Omega)}&\\ +|\varphi|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}+\|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)}\le c_{\mathrm{e}},&\end{aligned}\tag{A.13}$$

where the real number c_{e} only depends on Ω, the initial data $u_0(\cdot)$ and (as a decreasing function) on the parameter $\theta_{\mathcal{M}}$ characterizing the regularity of the mesh, defined by (A.2)*.*

Proof. First, we recall that, by a standard reordering of the summations, we have, for any function $u\in\mathrm{H}_{\mathcal{M}}(\Omega)$:

$$-\sum_{K\in\mathcal{M}}|K|\ u_K\ (\Delta_{\mathcal{M},\mathrm{D}}(u))_K=\|u\|^2_{1,\mathcal{M}},\qquad-\sum_{K\in\mathcal{M}}|K|\ u_K\ (\Delta_{\mathcal{M},\mathrm{N}}(u))_K=|u|^2_{1,\mathcal{M}}.$$

Multiplying by $2\,\delta t\,|K|\,u^{n+1}_K$ Equation (A.7)-(i), using the equality $2\,a\,(a-b)=a^2+(a-b)^2-b^2$, summing over each control volume of the mesh and using the first of the preceding identities yields, for $0\le n\le N-1$:

$$\begin{aligned}\|u^{n+1}\|^2_{L^2(\Omega)}+\|u^{n+1}-u^n\|^2_{L^2(\Omega)}-\|u^n\|^2_{L^2(\Omega)}+2\,\delta t\,\|u^{n+1}\|^2_{1,\mathcal{M}}&\\ +2\,\delta t\int_\Omega(u^{n+1})^5\,\mathrm{d}\boldsymbol{x}=2\,\delta t\int_\Omega\varphi^nu^{n+1}\,\mathrm{d}\boldsymbol{x}.&\end{aligned}$$

By Proposition A.2.1, u^{n+1} is non-negative and $\varphi_K^n \leq \bar{u}_0^4$, $\forall K \in \mathcal{M}$, where $\bar{u}_0$ stands for $\max_{K\in\mathcal{M}} u_K^0$ and is thus bounded by the L^∞-norm of on the initial data $u_0(\cdot)$. Therefore, we get:

$$\|u^{n+1}\|^2_{L^2(\Omega)} - \|u^n\|^2_{L^2(\Omega)} + 2\,\delta t\,\|u^{n+1}\|^2_{1,\mathcal{M}} \leq 2\,\delta t\,\bar{u}_0^4 \int_\Omega u^{n+1}\,\mathrm{d}\boldsymbol{x}.$$

By Cauchy-Schwarz' inequality, Young's inequality and the discrete Poincaré inequality (A.5), we thus obtain:

$$\|u^{n+1}\|^2_{L^2(\Omega)} - \|u^n\|^2_{L^2(\Omega)} + \delta t\,\|u^{n+1}\|^2_{1,\mathcal{M}} \leq \delta t\,|\Omega|\,\mathrm{diam}(\Omega)^2\,\bar{u}_0^8.$$

Summing from $n=0$ to $n=N-1$, we get:

$$\|u^N\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \delta t\,\|u^n\|^2_{1,\mathcal{M}} \leq T\,|\Omega|\,\mathrm{diam}(\Omega)^2\,\bar{u}_0^8 + \|u^0\|^2_{L^2(\Omega)}.$$

Since, by assumption, $u_0 \in \mathrm{H}^1_0(\Omega)$, the discrete H^1 norm of u^0, $\|u^0\|_{1,\mathcal{M}}$ is bounded by $c\|u_0\|_{\mathrm{H}^1(\Omega)}$ where the real number c only depends on Ω and, in a decreasing way, on the parameter $\theta_\mathcal{M}$ characterizing the regularity of the mesh (see e.g. Lemma 3.3 in [7]). Together with the preceding relation, this provides the control of the first term in (A.13).

We now turn to the estimate of $\|\partial_{t,\mathcal{D}}(u)\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_\mathcal{M})}$. Let v be a function of $\mathrm{H}_\mathcal{M}(\Omega)$; multiplying by $|K|\,v_K$ the first equation of the scheme (A.7)-(i) and summing over $K \in \mathcal{M}$, we get for $0 \leq n \leq N-1$:

$$\int_\Omega \partial_{t,\mathcal{D}}(u)^n\; v\,\mathrm{d}\boldsymbol{x} = -[u^{n+1}, v]_\mathcal{M} - \int_\Omega \left[(u^{n+1})^4 - \varphi^n\right]\, v\,\mathrm{d}\boldsymbol{x}.$$

By the fact that, as both $(u^{n+1})^4$ and φ^n are non-negative functions bounded by $\bar{u}_0^4$, the difference is itself bounded by $\bar{u}_0^4$, using the Cauchy-Schwarz inequality and the discrete Poincaré inequality (A.5), we get:

$$\int_\Omega \partial_{t,\mathcal{D}}(u)^n\; v\,\mathrm{d}\boldsymbol{x} \leq \Big[\|u^{n+1}\|_{1,\mathcal{M}} + \bar{u}_0^4\,|\Omega|^{1/2}\,\mathrm{diam}(\Omega)\Big]\,\|v\|_{1,\mathcal{M}},$$

and so:

$$\|\partial_{t,\mathcal{D}}(u)^n\|_{-1,\mathcal{M}} \leq \|u^{n+1}\|_{1,\mathcal{M}} + \bar{u}_0^4\,|\Omega|^{1/2}\,\mathrm{diam}(\Omega),$$

which, by the bound of $\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_\mathcal{M})}$, yields the control of the second term in (A.13).

As far as φ is concerned, the second equation of the scheme (A.7)-(ii) and the initialization (A.8) yields for $0 \leq n \leq N$:

$$\|\varphi^n\|^2_{L^2(\Omega)} + |\varphi^n|^2_{1,\mathcal{M}} \leq \int_\Omega \bar{u}_0^4\;\varphi^n\,\mathrm{d}\boldsymbol{x}.$$

Thus, by Young's inequality, we get:

$$\frac{1}{2}\,\|\varphi^n\|^2_{L^2(\Omega)} + |\varphi^n|^2_{1,\mathcal{M}} \leq \frac{1}{2}\,|\Omega|\,\bar{u}_0^8.$$

Multiplying by δt and summing over the time steps, this provides the estimates of $\|\varphi\|_{L^2((0,T)\times\Omega)}$ and $|\varphi|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_\mathcal{M})}$ we are searching for.

To obtain a control on $\partial_{t,\mathcal{D}}(\varphi)$, we need a sharper estimate on $\partial_{t,\mathcal{D}}(u)$. Our starting point is once again Equation (A.7)-(i), which we multiply this time by $|K|\;\partial_{t,\mathcal{D}}(u)^n$ before summing over $K \in \mathcal{M}$, to get for $0 \leq n \leq N-1$, once again by the identity $a^2 - b^2 \leq a^2 + (a-b)^2 - b^2 = 2\,a\,(a-b)$ and invoking the Cauchy-Schwarz inequality:

$$\|\partial_{t,\mathcal{D}}(u)^n\|^2_{L^2(\Omega)} + \frac{1}{2\,\delta t}\,\left(\|u^{n+1}\|^2_{1,\mathcal{M}} - \|u^n\|^2_{1,\mathcal{M}}\right) \leq |\Omega|^{1/2}\,\bar{u}_0^4\;\|\partial_{t,\mathcal{D}}(u)^n\|_{L^2(\Omega)},$$

so, by Young's inequality:

$$\|\partial_{t,\mathcal{D}}(u)^n\|^2_{L^2(\Omega)} + \frac{1}{\delta t}\,\left(\|u^{n+1}\|^2_{1,\mathcal{M}} - \|u^n\|^2_{1,\mathcal{M}}\right) \leq |\Omega|\,\bar{u}_0^8.$$

Multiplying by δt and summing over the time steps yields:

$$\|\partial_{t,\mathcal{D}}(u)\|^2_{L^2((0,T)\times\Omega)} + \|u^N\|^2_{1,\mathcal{M}} \leq |\Omega|\, T\, \bar{u}_0^8 + \|u^0\|^2_{1,\mathcal{M}}, \tag{A.14}$$

which provides an estimate for $\|\partial_{t,\mathcal{D}}(u)\|_{L^2((0,T)\times\Omega)}$. Taking now the difference of the second equation of the scheme (A.7)-(ii) at two consecutive time steps and using (A.8) for the first one, we obtain for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \partial_{t,\mathcal{D}}(\varphi)^n_K - (\Delta_{\mathcal{M},\mathrm{N}}(\partial_{t,\mathcal{D}}(\varphi)^n))_K = \frac{(u_K^{n+1})^4 - (u_K^n)^4}{\delta t}$$
$$= \left[(u_K^{n+1})^3 + (u_K^{n+1})^2\, u_K^n + u_K^{n+1}\, (u_K^n)^2 + (u_K^n)^3\right] \partial_{t,\mathcal{D}}(u)^n_K.$$

Multiplying by $|K|\ \partial_{t,\mathcal{D}}(\varphi)^n_K$ over each control volume of the mesh and summing yields:

$$\|\partial_{t,\mathcal{D}}(\varphi)^n\|^2_{L^2(\Omega)} + |\partial_{t,\mathcal{D}}(\varphi)^n|^2_{1,\mathcal{M}} \leq 4\bar{u}_0^3\ \|\partial_{t,\mathcal{D}}(u)^n\|_{L^2(\Omega)}\ \|\partial_{t,\mathcal{D}}(\varphi)^n\|_{L^2(\Omega)},$$

which, using Young's inequality, multiplying by δt and summing over the time steps yields the desired estimate for $\|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)}$, thanks to (A.14). ☐

We are now in position to prove the following existence and convergence result.

Theorem A.3.2. *Let* $(u^{(m)})_{m\in\mathbb{N}}$ *and* $(\varphi^{(m)})_{m\in\mathbb{N}}$ *be a sequence of solutions to* (A.7) *with a sequence of discretizations such that the space and time step,* $h^{(m)}$ *and* $\delta t^{(m)}$ *respectively, tends to zero. We suppose that the parameters* $\theta_{\mathcal{M}^{(m)}}$ *characterizing the regularity of the meshes of this sequence are bounded away from zero, i.e.* $\theta_{\mathcal{M}^{(m)}} \geq \theta > 0$, $\forall\, m \in \mathbb{N}$. *Then there exists a subsequence, still denoted by* $(u^{(m)})_{m\in\mathbb{N}}$ *and* $(\varphi^{(m)})_{m\in\mathbb{N}}$ *and two functions* $\tilde{u}$ *and* $\tilde{\varphi}$ *such that:*

1. $u^{(m)}$ *and* $\varphi^{(m)}$ *tends to* $\tilde{u}$ *and* $\tilde{\varphi}$ *respectively in* $\mathrm{L}^2((0,T)\times\Omega)$,
2. $\tilde{u}$ *and* $\tilde{\varphi}$ *are solution to the continuous problem* (A.1) *in the following weak sense:*
   $\tilde{u} \in \mathrm{L}^\infty((0,T)\times\Omega) \cap \mathrm{L}^2(0,T;\,\mathrm{H}^1_0(\Omega))$, $\tilde{\varphi} \in \mathrm{L}^\infty((0,T)\times\Omega) \cap \mathrm{L}^2(0,T;\,\mathrm{H}^1(\Omega))$
   and:

$$\left|\begin{array}{ll} \displaystyle\int_{0,T}\int_\Omega \left[-\frac{\partial\psi}{\partial t}\,\tilde{u} + \boldsymbol{\nabla}\tilde{u}\cdot\boldsymbol{\nabla}\psi + (\tilde{u}^4 - \tilde{\varphi})\,\psi\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \psi(x,0)\,u_0(x)\,\mathrm{d}\boldsymbol{x}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\psi \in \mathrm{C}^\infty_c([0,T)\times\Omega), & \\ \displaystyle\int_{0,T}\int_\Omega \left[(\tilde{\varphi} - \tilde{u}^4)\,\psi + \boldsymbol{\nabla}\tilde{\varphi}\cdot\boldsymbol{\nabla}\psi\right]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0, \qquad \forall\psi \in \mathrm{C}^\infty_c([0,T)\times\bar{\Omega}). & \end{array}\right. \tag{A.15}$$

Proof. The following estimates of the space translates can be found in [4], Lemma 9.3, p. 770 and Lemma 18.3, p. 851:

$$\forall v \in \mathrm{H}_\mathcal{M},\ \forall \eta \in \mathbb{R}^d,$$
$$\|\hat{v}(\cdot+\eta) - \hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \|v\|^2_{1,\mathcal{M}}\ |\eta|\ \left[|\eta| + c(\Omega)\,h\right],$$
$$\|\hat{v}(\cdot+\eta) - \hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq |\eta|\ \left[|v|^2_{1,\mathcal{M}}\ (|\eta| + 2\,h) + 2\,|\partial\Omega|\,\|v\|_{\mathrm{L}^\infty(\Omega)}\right],$$

where $\hat{v}$ stands for the extension by zero of v to $\mathbb{R}^d$ and the real number $c(\Omega)$ only depends on the domain.

For m given, let $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$ be the functions of $\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R})$ obtained by extending $u^{(m)}$ and $\varphi^{(m)}$ by 0 from $\Omega\times[0,T)$ to $\mathbb{R}^d\times\mathbb{R}$. The estimates of Proposition A.3.1 of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_\mathcal{M})}$ and $|\varphi^{(m)}|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_\mathcal{M})}$, together with the L^∞-bound for $\varphi^{(m)}$, thus allow to bound independently of m the space translates of $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$ in the $\mathrm{L}^2((0,T)\times\Omega)$-norm. In addition, Theorem A.a.2 applied with $\|\cdot\|_*$ equal to $\|\cdot\|_{1,\mathcal{M}}$ for $u^{(m)}$ and with $\|\cdot\|_*$ equal to $\|\cdot\|_{\mathrm{L}^2(\Omega)}$ for $\varphi^{(m)}$, together with the estimates of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_\mathcal{M})}$, $\|\partial_{t,\mathcal{D}}(u)^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^{-1}_\mathcal{M})}$, $\|\varphi\|_{L^2((0,T)\times\Omega)}$ and $\|\partial_{t,\mathcal{D}}(\varphi)\|_{L^2((0,T)\times\Omega)}$ of Proposition A.3.1 allows to bound the time translates of $\hat{u}^{(m)}$ and $\hat{\varphi}^{(m)}$, still independently of m and in the $\mathrm{L}^2((0,T)\times\Omega)$-norm. In addition, by Proposition A.2.1, the sequences $(\hat{u}^{(m)})_{m\in\mathbb{N}}$ and $(\hat{\varphi}^{(m)})_{m\in\mathbb{N}}$ are uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)$, and thus in $\mathrm{L}^2((0,T)\times\Omega)$. By Kolmogorov theorem (see e.g. [4], Theorem 14.1, p. 833), these sequences are relatively compact and strongly converge in $\mathrm{L}^2((0,T)\times\Omega)$ to, respectively, $\tilde u$ and $\tilde\varphi$. Moreover, the uniform bounds of $\|u^{(m)}\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal M})}$ and $|\varphi^{(m)}|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal M})}$ prove, by Theorem 14.2 and Theorem 14.3, p. 833 and p.834, in [4], that $\tilde u$ and $\tilde\varphi$ lie respectively in $\mathrm{L}^2(0,T;\,\mathrm{H}^1_0(\Omega))$ and $\mathrm{L}^2(0,T;\,\mathrm{H}^1(\Omega))$.

To prove that $\tilde u$ and $\tilde\varphi$ are solution to the continuous problem, it remains to prove that (A.15) holds. This proof is rather standard (see e.g. the proof of Theorem 18.1 pp. 858–862 in [4] for a similar, although more complicated, problem) and we only give here the main arguments. Let ψ be a function of $\mathrm{C}^\infty_c([0,T)\times\Omega)$. We define ψ^n_K by $\psi^n_K=\psi(x_K,t^n)$. Multiplying the first equation of (A.7) by $\delta t\,|K|\;\psi^{n+1}_K$ and summing up over the control volumes and the time steps, we get for any element of the sequence of discrete solutions:

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal M}\delta t\,|K|\;\psi^{n+1}_K\left[\frac{u^{n+1}_K-u^n_K}{\delta t}-(\Delta_{\mathcal M,\mathrm D}(u^{n+1}))_K+(u^{n+1}_K)^4-\varphi^n_K\right]$$
$$=T_1+T_2+T_3=0,$$

where, for enhanced readability, the superscript $^{(m)}$ has been omitted and:

$$\left|\begin{aligned}
T_1&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal M}|K|\;\psi^{n+1}_K\;\left[u^{n+1}_K-u^n_K\right],\\
T_2&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal M}-\delta t\,|K|\;\psi^{n+1}_K\;(\Delta_{\mathcal M,\mathrm D}(u^{n+1}))_K,\\
T_3&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal M}\delta t\,|K|\;\psi^{n+1}_K\;\left[(u^{n+1}_K)^4-\varphi^n_K\right].
\end{aligned}\right.$$

Reordering the summations and using the fact that $\psi(\cdot,T)=0$, we get for T_1:

$$T_1=-\sum_{K\in\mathcal M}|K|\;\psi^1_K\;u^0_K+\sum_{n=1}^{N-1}\sum_{K\in\mathcal M}|K|\;u^n_K\;\left[\psi^n_K-\psi^{n+1}_K\right].$$

The first term of the right hand side reads:

$$T_{1,1}=-\int_\Omega u_0(x)\,\psi(x,0)\,\mathrm d\boldsymbol x\;+\sum_{K\in\mathcal M}\int_K(u_0(x)-u^0_K)\,\psi(x,0)\,\mathrm d\boldsymbol x$$
$$+\sum_{K\in\mathcal M}|K|\;u^0_K\;\underbrace{\left[\frac{1}{|K|}\int_K\psi(x,0)\,\mathrm d\boldsymbol x-\psi(x_K,\delta t)\right]}_{R_\psi}.$$

On one hand, u^0 converges to u_0 in $\mathrm L^1\Omega$ and $\psi(\cdot,0)\in\mathrm C^\infty_c(\Omega)$, so the second term of $T_{1,1}$ tends to zero with h; on the other hand, since $u^0_K\le\bar u_0$, $\forall K\in\mathcal M$ and, from the regularity of ψ, $|R_\psi|\le c_\psi\;(\delta t+h)$, the third term of $T_{1,1}$ also tends to zero with δt and h. Let us now turn to the second term in the expression of T_1:

$$T_{1,2}=\sum_{n=1}^{N-1}\sum_{K\in\mathcal M}|K|\;u^n_K\;\left[\psi^n_K-\psi^{n+1}_K\right]=-\int_{\delta t}^T\int_\Omega u(x,t)\;\frac{\partial\psi}{\partial t}(x,t)\,\mathrm d\boldsymbol x\,\mathrm dt$$
$$+\sum_{n=1}^{N-1}\sum_{K\in\mathcal M}\delta t\,|K|\;u^n_K\;(R_\psi)^n_K$$

withr:

$$(R_\psi)^n_K=\frac{1}{\delta t\,|K|}\int_{t^n}^{t^{n+1}}\int_K\frac{\partial\psi}{\partial t}(x,t)\,\mathrm d\boldsymbol x\,\mathrm dt-\frac{\psi(x_K,t^{n+1})-\psi(x_K,t^n)}{\delta t},$$

and thus $|(R_\psi)^n_K|\le c_\psi\;(\delta t+h)$. Since $u^n_K\le\bar u_0$, $\forall K\in\mathcal M$ and $0\le n\le N$, we get:

$$T_1\to-\int_\Omega u_0(x)\,\psi(x,0)\,\mathrm d\boldsymbol x-\int_0^T\int_\Omega\tilde u(x,t)\;\frac{\partial\psi}{\partial t}(x,t)\,\mathrm d\boldsymbol x\,\mathrm dt\qquad\text{as}\quad m\to\infty.$$

Reordering the summations in T_2 and using the fact that $\psi(\cdot,T)=0$, we obtain:

$$\begin{aligned} T_2 &= \sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}} -\delta t\,|K|\; u_K^{n+1}\;(\Delta_{\mathcal{M},\mathrm{D}}(\psi^{n+1}))_K \\ &= -\int_{\delta t}^{T}\int_\Omega u(x,t)\;\Delta\psi(x,t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}} -\delta t\,|K|\; u_K^{n+1}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\;(R_\psi)_\sigma^{n+1}, \end{aligned}$$

where the residual term $(R_\psi)_\sigma^{n+1}$ is the difference of the mean value of $\boldsymbol{\nabla}\psi\cdot n$ over $\sigma\times(t^{n+1},t^{n+2})$ and its finite volume approximation. The fact that the second term in the right hand side of this relation tends to zero is thus a classical consequence of the control of $\|u\|_{\mathrm{L}^2(0,T;\,\mathrm{H}^1_{\mathcal{M}})}$ and the consistency of the diffusive fluxes $(R_\psi)_\sigma^{n+1}$ (see Theorem 9.1, pp. 772–776, in [4]) and yields, as $\tilde u$ is known to belong to $\mathrm{H}^1_0(\Omega)$:

$$T_2 \to \int_0^T\int_\Omega \boldsymbol{\nabla}\tilde u(x,t)\cdot\boldsymbol{\nabla}\psi(x,t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad \text{as}\quad m\to\infty.$$

Finally, T_3 reads:

$$\begin{aligned} T_3 = \int_{\delta t}^{T}\int_\Omega \psi(x,t)\;\big[u(x,t)^4-\varphi(x,t-\delta t)\big] \\ -\sum_{n=0}^{N-2}\sum_{K\in\mathcal{M}}\delta t\,|K|\;\big[(u_K^{n+1})^4-\varphi_K^n\big]\;(R_\psi)_K^{n+1} \end{aligned}$$

withr:

$$(R_\psi)_K^{n+1} = \frac{1}{\delta t\,|K|}\int_{t^{n+1}}^{t^{n+2}}\int_K \psi(x,t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \psi(x_K,t^{n+1}).$$

The second term tends to zero by the L^∞-estimates for u and φ and the regularity of ψ. Since $u^{(m)}$ tends to $\tilde u$ in $\mathrm{L}^2((0,T)\times\Omega)$ and is bounded in $\mathrm{L}^\infty((0,T)\times\Omega)$, $u^{(m)}$ converges to $\tilde u$ in $\mathrm{L}^2(0,T;\,\mathrm{L}^p(\Omega))$, for any $p\in[1,+\infty)$; in addition, from the time translates estimates, $\varphi^{(m)}(\cdot,\cdot-\delta t)$ also converge to $\tilde\varphi$. We thus get:

$$T_3 \to \int_0^T\int_\Omega \psi(x,t)\;\big[\tilde u(x,t)^4-\tilde\varphi(x,t)\big]\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad \text{as}\quad m\to\infty.$$

Gathering the results for T_1, T_2 and T_3, we obtain the first relation of (A.15). The second relation is obtained using the same arguments; the convergence of the diffusion term in case of Neumann boundary conditions poses an additional difficulty which is solved in Theorem 10.3, pp. 810–815, in [4]. □

The uniqueness of the solution to the problem under consideration is left beyond the scope of the present paper. Note however that such a result would imply, by a standard argument, the convergence of the whole sequence to the solution.

A.4 Conclusion

We propose in this paper a finite volume scheme for a problem capturing the essential difficulties of a simple radiative transfer model (the so-called $\mathbf{P}_1$ model), which enjoys the following properties: the discrete solution exists, is unique, and satisfies a discrete maximum principle; in addition, it converges (possibly up to the extraction of a subsequence) to a solution of the continuous problem, which yields, as a by-product, that such a solution indeed exists. For the proof of this latter result, we state and prove an abstract estimate allowing to bound the time translates of a finite volume discrete function, as a function of (possibly discrete) norms of the function itself and of its discrete time derivative; although this estimate is underlying in some already available analysis (see chapter IV in [4], [5] or [6]), this formulation is new and should be useful to tackle new problems. Variants of the presented numerical scheme are now successfully running for the modelling of radiative transfer in the ISIS free software [8] developed at IRSN and devoted to the simulation of fires in confined buildings (see [1]), as nuclear power plants.

A.a Estimation of time translates

The objective of this appendix is to state and prove an abstract result allowing to bound the time translates of a discrete solution. We begin by a technical lemma.

Lemma A.a.1. *Let $(t^n)_{0\leq n\leq N}$ be such that $t^0=0$, $t^n=n\delta t$, $t^N=T$, τ be a positive real number and $\chi^n_\tau:\ \mathbb{R}\to\mathbb{R}$ be the function defined by $\chi^n_\tau(t)=1$ if $t<t^n\leq t+\tau$ and $\chi^n_\tau(t)=0$ otherwise. Then, for any family of real numbers $(\alpha_n)_{n=1,N}$ and, respectively, for any real number t, we have the following identities:*

$$(i)\qquad \int_{\mathbb{R}}\left[\sum_{n=1}^{N}\alpha_n\chi^n_\tau(t)\right]\,\mathrm{d}t=\tau\sum_{n=1}^{N}\alpha_n,$$

$$(ii)\qquad \int_t^{t+\delta t}\left[\sum_{n=1}^{N}\chi^n_\tau(s)\right]\,\mathrm{d}s\leq\tau.$$

Proof. The function $\chi^n_\tau(t)$ is equal to one for $t\in[t^n-\tau,\ t^n)$, so we have:

$$\int_{\mathbb{R}}\left[\sum_{n=1}^{N}\alpha_n\chi^n_\tau(t)\right]\,\mathrm{d}t=\sum_{n=1}^{N}\alpha_n\int_{t^n-\tau}^{t^n}\,\mathrm{d}t.$$

To obtain the inequality (ii), we remark that $t\in[t^n-\tau,\ t^n)$ is equivalent to $t-t^n\in[-\tau,\ 0)$ and so $\chi^n_\tau(t)=1$ is equivalent to $\chi^0_\tau(t-t^n)=1$ (under the assumption $t^0=0$). We thus have:

$$\int_t^{t+\delta t}\left[\sum_{n=1}^{N}\chi^n_\tau(s)\right]\,\mathrm{d}s=\sum_{n=1}^{N}\int_{t-t^n}^{t-t^n+\delta t}\chi^0_\tau(s)\,\mathrm{d}s\leq\int_{\mathbb{R}}\chi^0_\tau(s)\,\mathrm{d}s=\tau.$$

□

We now introduce some notations. Let $\mathrm{H}_{\mathcal{M}}(\Omega)$ and $\mathrm{H}_{\mathcal{D}}$ be the discrete functional spaces introduced in section A.1 and A.3 respectively. We suppose given a norm $\|\cdot\|_*$ on $\mathrm{H}_{\mathcal{M}}(\Omega)$, over which we also define the dual norm $\|\cdot\|^*$ with respect to the L^2-inner product:

$$\forall u\in\mathrm{H}_{\mathcal{M}}(\Omega),\qquad \|u\|^*\stackrel{\text{def}}{=}\sup_{v\in\mathrm{H}_{\mathcal{M}}(\Omega),\,v\neq0}\frac{\displaystyle\int_\Omega u\,v\,\mathrm{d}\boldsymbol{x}}{\|v\|_*}.$$

These two spatial norms may be associated to a corresponding norm on $\mathrm{H}_{\mathcal{D}}$ as follows:

$$\forall u\in\mathrm{H}_{\mathcal{D}},\ u=(u^n)_{0\leq n\leq N},\qquad \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_*)}=\sum_{n=0}^{N}\delta t\ \|u^n\|_*^2,$$

$$\text{and}\qquad \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*)}=\sum_{n=0}^{N-1}\delta t\ \|u^n\|^{*2}.$$

We are now in position to state the following result.

Theorem A.a.2. *Let u be a function of $\mathrm{H}_{\mathcal{D}}$ and τ a real number. We denote by $\hat{u}$ the extension by zero of u to $\mathbb{R}^d\times\mathbb{R}$. Then we have:*

$$\|\hat{u}(\cdot,\cdot+\tau)-\hat{u}(\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R})}\leq\tau\ \Big[2\ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_*)}+\frac{1}{2}\ \|\partial_{t,\mathcal{D}}(u)\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*)}+2\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}\Big].$$

Proof. Let u be a function of $\mathrm{H}_\mathcal{D}$ and $t \in \mathbb{R}$. Let τ be a real number that we suppose positive. The following identity holds:

$$\hat{u}(\cdot, t+\tau) - \hat{u}(\cdot, t) = \chi^0_\tau(t)\ u^0 + \sum_{n=1}^{N-1} \chi^n_\tau(t) \left[u^n - u^{n-1}\right] - \chi^N_\tau(t)\ u^{N-1}.$$

For $s \in \mathbb{R}$ we define $n(s)$ by: $n(s) = -1$ if $s < 0$, $n(s)$ is the index such that $t^{n(s)} \leq s < t^{n(s)+1}$ for $0 \leq s < t^N$, $n(s) = N+1$ for $t \geq t^N$. Let $n_0(t)$ and $n_1(t)$ be given by $n_0(t) = n(t)$, $n_1(t) = n(t+\tau)$. We adopt the convention $u^{-1} = u^N = 0$. With this notation, we have for $u(\cdot, t+\tau) - u(\cdot, t)$ the following equivalent expression:

$$u(\cdot, t+\tau) - u(\cdot, t) = u^{n_1(t)} - u^{n_0(t)},$$

and thus:

$$\int_\Omega \left[u(x, t+\tau) - u(x,t)\right]^2\ \mathrm{d}\boldsymbol{x} = \\ \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ \left[\chi^0_\tau(t)\ u^0 + \sum_{n=1}^{N-1} \chi^n_\tau(t) \left[u^n - u^{n-1}\right] - \chi^N_\tau(t)\ u^{N-1}\right]\ \mathrm{d}\boldsymbol{x}.$$

Developping, we get:

$$\int_\Omega \left[u(x, t+\tau) - u(x,t)\right]^2\ \mathrm{d}\boldsymbol{x} = T_1(t) + T_2(t) + T_3(t)$$

withr:

$$\left|\begin{aligned} & T_1(t) = \chi^0_\tau(t) \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ u^0\ \mathrm{d}\boldsymbol{x}, \\ & T_2(t) = \sum_{n=1}^{N-1} \chi^n_\tau(t) \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right] \left[u^n - u^{n-1}\right]\ \mathrm{d}\boldsymbol{x}, \\ & T_3(t) = -\chi^N_\tau(t) \int_\Omega \left[u^{n_1(t)} - u^{n_0(t)}\right]\ u^{N-1}\ \mathrm{d}\boldsymbol{x}. \end{aligned}\right.$$

We first estimate the integral of $T_1(t)$ over $\mathbb{R}$. Since $\chi^0_\tau(t)$ is equal to 1 in the interval $[-\tau, 0)$ and 0 elsewhere, and since $u^{n_0(t)} = 0$ for any negative t, we get:

$$\int_\mathbb{R} T_1(t)\ \mathrm{d}t = \int_{-\tau}^0 \int_\Omega u^{n_1(t)}\ u^0\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \leq \tau\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}.$$

By the same arguments, we get the same bound for the integral of $T_3(t)$:

$$\int_\mathbb{R} T_3(t)\ \mathrm{d}t \leq \tau\ \|u\|^2_{\mathrm{L}^\infty(0,T;\,\mathrm{L}^2(\Omega))}.$$

From the definition of the $\|\cdot\|^*$ norm, we get:

$$T_2(t) \leq \delta t \sum_{n=1}^{N-1} \chi^n_\tau(t) \quad \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^*\ \|u^{n_1(t)} - u^{n_0(t)}\|_*,$$

and thus, by Young's inequality:

$$T_2(t) \leq \delta t \sum_{n=1}^{N-1} \chi^n_\tau(t) \quad \left[\frac{1}{2}\ \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^{*2} + \|u^{n_0(t)}\|^2_* + \|u^{n_1(t)}\|^2_*\right].$$

Integrating over the time, we get:

$$\int_\mathbb{R} T_2(t)\ \mathrm{d}t \leq T_{2,1} + T_{2,2} + T_{3,3},$$

where the term $T_{2,1}$ reads and, by Lemma A.a.1 (Relation (i)), satisfies:

$$T_{2,1} = \frac{\delta t}{2} \int_{\mathbb{R}} \sum_{n=1}^{N-1} \chi_\tau^n(t) \quad \|\partial_{t,\mathcal{D}}(u)^{n-1}\|^{*2} \,\mathrm{d}t \quad = \frac{\tau}{2} \sum_{n=0}^{N-2} \delta t \ \|\partial_{t,\mathcal{D}}(u)^n\|^{*2}$$
$$\leq \frac{\tau}{2} \ \|\partial_{t,\mathcal{D}}(u)\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}^*)}.$$

Since $u^{n_0(t)} = u^m$ for $t^m \leq t < t^{m+1}$, the term $T_{2,2}$ reads and satisfies, once again by Lemma A.a.1 (Relation (ii)):

$$T_{2,2} = \delta t \sum_{m=0}^{N-1} \left[\int_{t^m}^{t^{m+1}} \sum_{n=1}^{N-1} \chi_\tau^n(t) \,\mathrm{d}t \right] \|u^m\|_*^2 \leq \tau \sum_{m=0}^{N-1} \delta t \ \|u^m\|_*^2 = \tau \ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_*)}.$$

Finally, $u^{n_1(t)} = u^m$ for $t^m - \tau \leq t < t^{m+1} - \tau$, and thus, by the same argument:

$$T_{2,3} = \delta t \sum_{m=0}^{N-1} \left[\int_{t^m-\tau}^{t^{m+1}-\tau} \sum_{n=1}^{N-1} \chi_\tau^n(t) \,\mathrm{d}t \right] \|u^m\|_*^2 \leq \tau \sum_{m=0}^{N-1} \delta t \ \|u^m\|_*^2 = \tau \ \|u\|^2_{\mathrm{L}^2(0,T;\,\mathrm{H}_*)}.$$

This concludes the proof for positive τ. The case of negative τ follows by symmetry. □

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Annexe B

A finite volume stability result for the convection operator in compressible flows
...and some finite element applications

A FINITE VOLUME STABILITY RESULT FOR THE CONVECTION OPERATOR IN COMPRESSIBLE FLOWS ...AND SOME FINITE ELEMENT APPLICATIONS

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Abstract.In this paper, we build a L^2–stable discretization of the non-linear convection term in Navier-Stokes equations for non-divergence-free flows, for non-conforming low order Stokes finite elements. This discrete operator is obtained by a finite volume technique, and its stability relies on a result interesting for its own sake: the L^2–stability of the natural finite volume convection operator in compressible flows, under some compatibility condition with the discrete mass balance. Then, this analysis is used to derive a boundary condition to cope with physical situations where the velocity cannot be prescribed on inflow parts of the boundary of the computational domain. We finally collect these ingredients in a pressure correction scheme for low Mach number flows, and assess the capability of the resulting algorithm to compute a natural convection flow with artificial (open) boundaries.

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Introduction

Let ρ and $\boldsymbol{u}$ be a scalar and a vector smooth function respectively, defined over a domain Ω of $\mathbb{R}^d$, $d = 2$ or $d = 3$, and such that the following identity holds in Ω:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{B.1}$$

Let z be a smooth scalar function defined over Ω. Then the following stability identity is known to hold:

$$\int_\Omega \left[\frac{\partial \rho z}{\partial t} + \nabla \cdot (\rho z \boldsymbol{u}) \right] z = \frac{1}{2} \frac{d}{dt} \int_\Omega \rho z^2 + \frac{1}{2} \int_{\partial\Omega} \rho z^2 \boldsymbol{u} \cdot \boldsymbol{n} \tag{B.2}$$

Where ρ stands for the density and $\boldsymbol{u}$ for the velocity, equation (B.1) is the usual mass balance in a variable density flow. Choosing for z a component of the velocity, equation (B.2) yields the central argument of the kinetic energy conservation theorem.

In this paper, we first derive a finite volume analogue of relation (B.2); the statement of this stability estimate is the object of section B.1. Then, in section B.2, we show how this result may be used to build a L^2-stable convection operator for the Rannacher-Turek [8] or Crouzeix-Raviart [4] low order non-conforming Stokes finite elements, switching for this term from the finite element discretization to a finite volume approximation based on a dual mesh; note that a similar technique is implemented in [1, 6] for the solution of convection-diffusion type equations (so with a known continuous velocity field), however for a different purpose, namely to satisfy a discrete maximum principle. This discretization is applied to solve the balance equations of the asymptotic model for low Mach number flows, which reads:

$$\left|\begin{aligned} & \frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p - \nabla \cdot \tau(\boldsymbol{u}) = \boldsymbol{f}_v \\ & \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \end{aligned}\right. \tag{B.3}$$

where p stands for the pressure, $\tau(\boldsymbol{u})$ is the shear stress tensor and $\boldsymbol{f}_v$ is a forcing term. The density ρ is supposed to depend on a state variable of the fluid (*e.g.* the temperature, the composition... but not the pressure) which is solution to an additional balance equation. For this problem, we show in particular how the necessity to control the terms which arise in the kinetic energy balance under the form of boundary integrals, both from the second term at the right-hand side of (B.2) and from the integration by parts of the divergence of the stress tensor, suggests a boundary condition for artificial boundaries. This latter has been implemented in the ISIS code developed at IRSN, with the purpose to allow this open-source CFD tool to cope with the simulation of fires in an open atmosphere; it is assessed in section B.4 against a natural convection model problem.

B.1 A finite volume result

Let a finite volume admissible mesh $\mathcal{M}$ (in the sense of [5], Chapter 3) of the computational domain Ω be given. This mesh is composed of a family $\mathcal{M}$ of control volumes, which are convex disjoint polygons ($d = 2$) or polyhedrons ($d = 3$) included in Ω and such that $\bar{\Omega} = \bigcup_{K \in \mathcal{M}} \bar{K}$. For each neighbouring control volume L of $K \in \mathcal{M}$, $\sigma = K|L$ denotes the common edge or face of K and L. The sets $\mathcal{E}_{\rm int}$, $\mathcal{E}_{\rm ext}$ and $\mathcal{E}(K)$ stand respectively for the internal edges or faces (*i.e.* separating two control volumes), the external ones (*i.e.* included in the boundary) and the edges or faces of the control volume K. By $|K|$ and $|\sigma|$, we denote hereafter the $d-$ and $(d-1)-$ dimensional measures of $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, respectively.

Let $(\rho^*_K)_{K \in \mathcal{M}}$ and $(\rho_K)_{K \in \mathcal{M}}$ be two families of positive real numbers satisfying the following set of equations:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\,(\rho_K - \rho^*_K) + \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K} = 0 \tag{B.4}$$

where $F_{\sigma,K}$ is a quantity associated to the edge σ and to the control volume K; we suppose that, for any internal edge $\sigma = K|L$, $F_{\sigma,K} = -F_{\sigma,L}$. Equation (B.4) may be seen as the finite-volume counterpart of the continuous mass balance (B.1).

Let $(z^*_K)_{K \in \mathcal{M}}$ and $(z_K)_{K \in \mathcal{M}}$ be two families of real numbers. For any internal edge or face $\sigma = K|L$, we define z_σ either by $z_\sigma = \frac{1}{2}(z_K + z_L)$, or by $z_\sigma = z_K$ if $F_{\sigma,K} \geq 0$ and $z_\sigma = z_L$ otherwise. The first choice is usually referred to as the "centered choice", the second one as "the upwind choice" with respect to the quantity $F_{\sigma,K}$. For an external edge or face, if $F_{\sigma,K} \geq 0$, we suppose that $z_\sigma = z_K$ (*i.e.* that the upwind choice is made, which seems to be the only natural possibility in this case), and if $F_{\sigma,K} \leq 0$, we suppose that z_σ is given by a relation which we do not need to precise for the moment. Then we can state the following stability result.

Theorem B.1.1 (Stability of the convection operator)**.** *With the above definitions, the following stability estimate holds, for both the centered or upwind choice for the quantities z_σ:*

$$\begin{aligned} \sum_{K \in \mathcal{M}} z_K \left[\frac{|K|}{\delta t}\,(\rho_K\, z_K - \rho^*_K\, z^*_K) + \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K}\, z_\sigma \right] \geq \\ \frac{1}{2} \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t}\, \left[\rho_K\, z_K^2 - \rho^*_K\, {z^*_K}^2 \right] + \frac{1}{2} \sum_{\substack{\sigma \in \mathcal{E}_{\rm ext} \\ (\sigma \in \mathcal{E}(K))}} F_{\sigma,K}\, z_\sigma^2 \end{aligned} \tag{B.5}$$

In the case of a velocity vanishing on the boundary of the computational domain, a proof of this result can be found in [7].

B.2 A convection operator for low-order non-conforming finite elements

We now turn to the discretization of Navier-Stokes equations (B.3) by a low-order mixed finite element method.

B.2.1 Discretization spaces

We now suppose that the control volumes are either convex quadrilaterals ($d = 2$), hexahedra ($d = 3$) or simplices. In the first case, the spatial discretization relies on the so-called "rotated bilinear element"/P_0 introduced by Rannacher and Turek [8] (RT in the following); for simplicial meshes, the Crouzeix-Raviart element [4] (CR in the following) is used. The reference element $\widehat{K}$ for the RT element is the unit d-cube (with edges parallel to the coordinate axes); the discrete functional space on $\widehat{K}$ is $\tilde{Q}_1(\widehat{K})^d$, where $\tilde{Q}_1(\widehat{K})$ is defined as follows:

$$\tilde{Q}_1(\widehat{K}) = \operatorname{span}\left\{1,\ (x_i)_{i=1,\ldots,d},\ (x_i^2 - x_{i+1}^2)_{i=1,\ldots,d-1}\right\}$$

The reference element for the CR element is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. For both velocity elements used here, the degrees of freedom are determined by the following set of nodal functionals on the discrete velocity space:

$$\{\varphi_{\sigma,i},\ \sigma \in \mathcal{E}(K),\ i = 1,\ldots,d\}, \qquad \varphi_{\sigma,i}(\boldsymbol{v}) = |\sigma|^{-1} \int_\sigma \boldsymbol{v}_i \,\mathrm{d}\gamma$$

The mapping from the reference element to the actual one is, for the RT element, the standard Q_1 mapping and, for the CR element, the standard affine mapping. Finally, in both cases, the continuity of the average value of discrete velocities (*i.e.* , for a discrete velocity field $\boldsymbol{v}$, $\varphi_{\sigma,i}(\boldsymbol{v}),\ 1 \leq i \leq d$) across each edge or face of the mesh is required, and, as usual in finite elements methods, Dirichlet conditions are built-in in the approximation space, thus the discrete space $\boldsymbol{W}_h$ is defined as follows:

$$\begin{aligned} \boldsymbol{W}_h = \quad & \{\ \boldsymbol{v}_h \in L^2(\Omega)^d\ :\ \boldsymbol{v}_h|_K \in W(K)^d,\ \forall K \in \mathcal{M}; \\ & \varphi_{\sigma,i}(\boldsymbol{v}_h) \text{ continuous across each edge } \sigma \in \mathcal{E}_{\rm int},\ 1 \leq i \leq d\,; \\ & \varphi_{\sigma,i}(\boldsymbol{v}_h) =\ |\sigma|^{-1} \int_\sigma \boldsymbol{u}_{{\rm D},i} \,\mathrm{d}\gamma,\ \forall \sigma \in \mathcal{E}_{\rm ext,D},\ 1 \leq i \leq d\ \} \end{aligned}$$

where $W(K)$ is the discrete functions space on K, $\mathcal{E}_{\rm ext,D}$ is the set of the external edges included in the part of the boundary where the velocity is prescribed and $\boldsymbol{u}_{\rm D}$ is this prescribed velocity. From this definition, each velocity degree of freedom can be associated to an element edge. Hence, the set of velocity degrees of freedom may be written as $\{\boldsymbol{v}_{\sigma,i},\ \sigma \in \mathcal{E} \setminus \mathcal{E}_{\rm ext,D},\ 1 \leq i \leq d\}$. We define $\boldsymbol{v}_\sigma = \sum_{i=1}^d \boldsymbol{v}_{\sigma,i}\, e^{(i)}$ where $e^{(i)}$ is the i^{th} vector of the canonical basis of $\mathbb{R}^d$.

For both RT and CR discretizations, the pressure is approximated by piecewise constant functions. The same approximation is used for the density.

B.2.2 A convection operator

The natural finite element method for RT and CR elements cannot be L^2-stable; indeed, the derivation of a stability estimate of the form of (B.2) involves integrations by parts, which, because of the non-conformity of the discretization, makes uncontrolled jumps across the edges ($d = 2$) or faces ($d = 3$) of the elements appear. We thus approximate both the unsteady term (*i.e.* $\partial \rho \boldsymbol{u}/\partial t$) and the convection term (*i.e.* $\nabla \cdot \rho \boldsymbol{u} \otimes \boldsymbol{u}$) by a finite volume discretization, using for control volumes a dual mesh. From the definition of the velocity degrees of freedom $\boldsymbol{v}_\sigma$, a control volume for each of these latter must be associated to an edge or a face σ. For $\sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L$, this dual cell is defined as the union of the two cones of common basis σ and vertices x_K and x_L respectively, where x_K (resp. x_L) is the mass center of K (resp. L), see figure B.1; for $\sigma \in \mathcal{E}_{\rm ext} \setminus \mathcal{E}_{\rm ext,D}$, the cell is restricted to the cone included in the adjacent primal control volume. For each considered σ, the corresponding dual cell is denoted by D_σ and called in the following the "diamond cell associated to σ".

For simplicial or parallelepipedic meshes, an important property is that $|D_\sigma|$, the measure of D_σ, is also the integral over Ω of the shape function associated to σ, which shows that the definition of this cell is in some sense consistent with the results of a classical mass lumping of the unsteady term. This is the discretization used here.

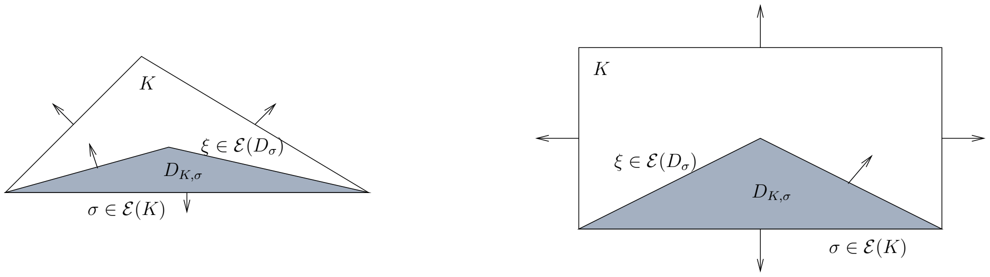

Figure B.1: Diamond-cells for the Crouzeix-Raviart and Rannacher-Turek element.

Making use of a finite volume technique for the term $\nabla \cdot \rho \boldsymbol{u} \otimes \boldsymbol{u}$ with the goal of applying theorem B.1.1 raises the problem to approximate the fluxes on the edges of the diamond cells in such a way that the discrete mass balance (B.4) holds. Indeed, as pressure discrete functions are piecewise constant over the primal cells, the mixed finite element formulation yields a finite-volume-like discrete mass balance based on the primal mesh, and not on the dual one. For the CR element, it may be seen that evaluating the mass flux on the boundary of the diamond cell ∂D_σ from the mass fluxes at the edges or faces of the primal mesh through the finite element expansion makes the mass balance hold on D_σ also. The proof of this elementary result relies on the fact that the divergence of a discrete velocity is constant over each primal cell. This result is extended to RT elements on parallelepidedic meshes by designing a specific interpolation, such that the divergence of the reconstructed mass flux field is also constant mesh-by-mesh [2].

B.3 A pressure correction scheme for low Mach number flows with open boundaries

On the basis of the preceding developments, we now derive a pressure correction scheme for the solution of system (B.3). The first equation, usually referred to as the velocity prediction step, consists in solving for a (non-divergence free) tentative velocity the momentum balance equation with the explicit pressure (*i.e.* the pressure at the previous time step); the convection term is linearized, by taking the explicit velocity as advective field. In variational form, this discrete equation consists in searching $\tilde{\boldsymbol{u}}^{n+1} \in \boldsymbol{W}_h$ such that, $\forall \boldsymbol{v} \in \boldsymbol{W}_h$:

$$\begin{aligned} \frac{1}{\delta t}(\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1}\boldsymbol{u}^n, \boldsymbol{v})_h + (\boldsymbol{\nabla}\cdot_h \rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n, \boldsymbol{v})_h \\ + a(\tilde{\boldsymbol{u}}^{n+1}, \boldsymbol{v}) + b(p^n, \boldsymbol{v}) = (\boldsymbol{f}^{n+1}, \boldsymbol{v})_h \end{aligned} \tag{B.6}$$

In this relation, the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ stand for the usual finite element discretizations of the viscous dissipation and the pressure gradient term, respectively, and the right hand side $\boldsymbol{f}$ gathers the effects of the forcing term and of the non-homogeneous Dirichlet boundary conditions. The notation $(\cdot,\cdot)_h$ stands for a discrete L^2 inner product defined by $(\boldsymbol{v}, \boldsymbol{w})_h = \sum_{\sigma \in \mathcal{E} \setminus \mathcal{E}_{\mathrm{ext,D}}} |D_\sigma| \; \boldsymbol{v}_\sigma \cdot \boldsymbol{w}_\sigma, \; \forall \boldsymbol{v} \in \boldsymbol{W}_h, \; \forall \boldsymbol{w} \in \boldsymbol{W}_h$. The discrete divergence operator is defined as described in the previous section:

$$(\boldsymbol{\nabla}\cdot_h \rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n)_\sigma = \frac{1}{|D_\sigma|} \sum_{\substack{\varepsilon \in \mathcal{E}(D_\sigma) \\ \varepsilon \in \mathcal{E} \setminus \mathcal{E}_{\mathrm{ext,D}}}} |\varepsilon| \; (\rho^n \boldsymbol{u}^n)_\varepsilon \; (\tilde{\boldsymbol{u}}^{n+1})_\varepsilon$$

where $|\varepsilon|$ stands for the measure of a face or edge ε of D_σ, and the centered choice is made for the approximation of $\tilde{\boldsymbol{u}}_\varepsilon^{n+1}$ on internal bounds; the external edges or faces of the dual mesh are also the external edges or faces of the primal one, and the approximation $\tilde{\boldsymbol{u}}_\varepsilon^{n+1} = \tilde{\boldsymbol{u}}_\sigma^{n+1}$ is thus natural in this case. The quantities $(\rho^n \boldsymbol{u}^n)_\varepsilon$ are obtained, by the interpolation previously described, from the mass fluxes appearing in the discrete mass balance at the previous time-step, because the mass balance at the current one is not solved at this stage of the algorithm: this is the reason of the time-shift of the density in this prediction step (B.6).

The second step of the algorithm is a standard algebraic projection step, and is not detailed here.

We now turn to the derivation of an artificial boundary condition. The question that we address is the following one: what could be a suitable condition for the inflow boundaries where the velocity is not prescribed? A part of the answer may come from energy estimates: indeed, it seems reasonable to require from this boundary condition not to lead to an unstable problem. A such energy estimate is obtained by taking $\boldsymbol{v} = \tilde{\boldsymbol{u}}^{n+1}$ in equation (B.6) (see [7] for this calculation), which yields, using the stability of the discrete convection operator (theorem B.1.1):

$$\frac{1}{2}\,(\rho^n \tilde{\boldsymbol{u}}^{n+1}, \tilde{\boldsymbol{u}}^{n+1})_h + \delta t\, T_{\rm visc}^{n+1} + \delta t\, T_{\rm pres}^{n+1} \leq \frac{1}{2}\,(\rho^{n-1} \boldsymbol{u}^n, \boldsymbol{u}^n)_h + \delta t\, T_{\rm D}^{n+1} + \delta t\, T_{\partial\Omega}^{n+1}$$

The first terms in the left- and right-hand sides are the discrete kinetic energy at time $t = t^{n+1}$ and $t = t^n$ respectively. Combining this estimate with additional bounds derived from the projection step, the pressure work term $T_{\rm pres}^{n+1}$ would more or less cancel if the flow was incompressible and would provide a control of the discrete time derivative of the elastic potential in the compressible case [7]. In the present case, this term is unfortunately not controlled, because the low Mach number model does not appear to be energetically consistent; we do not develop this point further here. The term $T_{\rm D}^{n+1}$ represents the contribution of the forcing term and of the non-homogeneous Dirichlet conditions collected in $\boldsymbol{f}^{n+1}$ and is obtained, as usual, by absorbing the contribution of the test function $\tilde{\boldsymbol{u}}^{n+1}$ to the inner product $(\boldsymbol{f}^{n+1}, \tilde{\boldsymbol{u}}^{n+1})_h$ in the viscous dissipation term $T_{\rm visc}^{n+1}$. By theorem B.1.1, we get for the last term (recall that the boundary edges or faces are the same for the primal and the dual cells):

$$T_{\partial\Omega}^{n+1} \leq \sum_{\sigma \in \mathcal{E}_{\rm ext} \setminus \mathcal{E}_{\rm ext,D}} -\frac{1}{2}\,|\sigma|\,(\rho^n \boldsymbol{u}^n)_\sigma\,|(\tilde{\boldsymbol{u}}^{n+1})_\sigma|^2 + \int_\sigma \big[\tau(\tilde{\boldsymbol{u}}^{n+1}) \cdot \boldsymbol{n}_\sigma - p^n\, \boldsymbol{n}_\sigma\big] \cdot \tilde{\boldsymbol{u}}^{n+1}$$

The scheme thus will be stable if this term can be controlled by the boundary condition, which may be obtained by replacing in the variational formulation (B.6), for the inflow edges or faces where the velocity is not prescribed, the terms multiplying the test function $\boldsymbol{v}$ in the following expression:

$$-\frac{1}{2}\,|\sigma|\,(\rho^n \boldsymbol{u}^n)_\sigma\,(\tilde{\boldsymbol{u}}^{n+1})_\sigma \cdot \boldsymbol{v}_\sigma + \int_\sigma \big[\tau(\tilde{\boldsymbol{u}}^{n+1}) \cdot \boldsymbol{n}_\sigma - p^n\, \boldsymbol{n}_\sigma\big] \cdot \tilde{\boldsymbol{u}}^{n+1}$$

by a known quantity. This is consistent with the following continuous boundary condition:

$$-\frac{1}{2}\,\rho\, \boldsymbol{u} \cdot \boldsymbol{n}_\sigma\ \boldsymbol{u} + \tau(\tilde{\boldsymbol{u}}) \cdot \boldsymbol{n}_\sigma - p\, \boldsymbol{n}_\sigma = \boldsymbol{f}_{\partial\Omega} \tag{B.7}$$

where the field $\boldsymbol{f}_{\partial\Omega}$, defined on $\partial\Omega$, is a part of the data of the problem. For incompressible flows, a theoretical study of the Navier-Stokes problem complemented with this condition can be found in [3, chapter V]; the conclusion is that the problem is well-posed. Note that this operation may be realised very simply in practice: for a concerned edge or face σ, the integrals involving the stress tensor and the pressure are not computed, the convection term is divided by 2 and the integral of $\boldsymbol{f}_{\partial\Omega} \cdot \boldsymbol{v}$ over σ is added.

B.4 Numerical test: a natural convection flow with open boundaries

To assess the behaviour of the presented scheme, we address a natural convection flow with artificial boundary conditions. For this test, system (B.3) is complemented by a standard energy balance, *i.e.* a linear convection-diffusion equation for the temperature, which is solved by an usual finite volume method.

The geometry of the computational domain is sketched on figure B.2. The boundary conditions are the following ones: on $\partial\Omega_{\rm D}$, the velocity is set to zero and the temperature is fixed to $T = 900°\,C$; on $\partial\Omega_{\rm S_1}$ and $\partial\Omega_{\rm S_2}$, the normal velocity is set to zero and a free slip is allowed, while the normal gradient of the

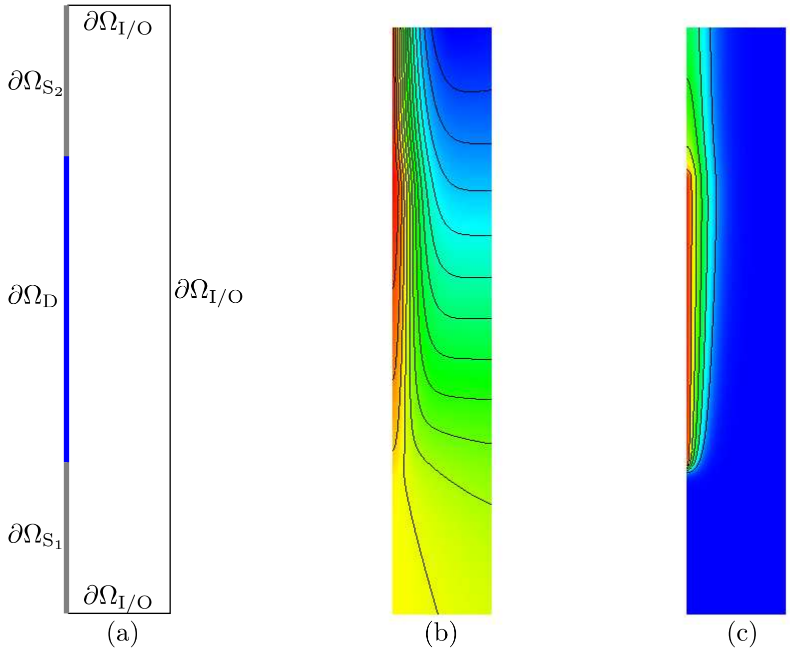

Figure B.2: (a) computational domain, (b) streamlines and (c) isovalues of the temperature ($T = 400°\,C$, $T = 500°\,C$, $T = 600°\,C$, $T = 700°\,C$ and $T = 900°\,C$)

.

temperature is set to zero; on the outflow part of $\partial\Omega_{I/O}$, a zero traction ($\tau(\boldsymbol{u}) \cdot \boldsymbol{n} - p\,\boldsymbol{n} = 0$) and a zero normal temperature gradient are imposed; on the inflow part of $\partial\Omega_{I/O}$, the artificial boundary condition (B.7) developed in the previous section is applied with $\boldsymbol{f}_{\partial\Omega} = 0$ while the temperature is set to $T = 300°\,C$. Note that the partition of $\partial\Omega_{I/O}$ in an inflow and outflow part is determined by the computation itself.

The fluid obeys the ideal gas law, with a constant equal to $R = 287$ and a constant pressure of $101325\,Pa$. The viscosity is fixed at the value of $1.68\,10^{-5}\,Pa.s$, the specific heat capacity under constant pressure is given by $c_p = R\gamma/(\gamma - 1)$ with $\gamma = 1.4$ and the Prandtl number is equal to 0.7. The width of the domain is $l = 0.01\,m$ and the height h is adjusted in such a way that the Rayleigh number, based on the height of the heated part of the boundary, is equal to 10^6 (so $h = 0.062\,m$).

Results obtained with a 58×248 regular grid are sketched on figure B.2. The steady state is obtained through a transient, starting from the initial condition $\boldsymbol{u} = 0$ and $T = 300°\,C$. The flow enters the domain on almost the whole part of $\partial\Omega_{I/O}$, except in the left side of the top boundary.

In addition, computations with a Rayleigh number of 10^7 and 10^8 were performed. No instability was seen during any of these runs. The essential effect observed when raising the Rayleigh number is a shrinking of the velocity and temperature boundary layer near the left boundary. Finally, we conducted computations with a larger domain; the obtained velocity and temperature profiles remain in remarkable agreement with the initial results, which shows that this boundary condition does not perturb the flow in the zone of interest.

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Schémas numériques pour les modèles de turbulence statistiques en un point

Résumé : Les modèles de turbulence de type Navier–Stokes en moyenne de Reynolds (RANS) au premier ordre sont étudiés dans cette thèse. Ils sont constitués des équations de Navier–Stokes, auxquelles on adjoint un système d'équations de bilan pour des échelles scalaires caractéristiques de la turbulence. L'évaluation de celles-ci permet, grâce à une relation algébrique, de calculer une viscosité additionnelle dite "turbulente", modélisant la contribution de l'agitation turbulente dans les équations de Navier–Stokes. Les problèmes d'analyse numérique abordés se placent dans le contexte d'un algorithme à pas fractionnaire constitué d'une approximation, sur un maillage régulier, des équations de Navier–Stokes par éléments finis non-conformes de Crouzeix–Raviart, ainsi que d'un ensemble d'équations de bilan de la turbulence de type convection–diffusion, discrétisées par la méthode de volumes finis standard.
Un schéma numérique basé sur une discrétisation de volumes finis, permettant de préserver la positivité des échelles turbulentes telles que l'énergie cinétique turbulente (k) et son taux de dissipation (ε), est ainsi proposé dans le cas des modèles $k - \varepsilon$ standard, $k - \varepsilon$ RNG et leur extension $k - \varepsilon - \overline{v^2} - f$.
La convergence du schéma numérique proposé est ensuite étudiée sur un problème modèle constitué des équations de Stokes incompressibles et d'une équation de convection–diffusion stationnaires, couplées par les viscosités et le terme de production turbulente. Il permet d'aborder la difficulté principale de l'analyse d'un tel problème : l'expression du terme de production turbulente amène à considérer, pour les équations de bilan de la turbulence, un problème de convection–diffusion avec second membre appartenant à L^1.
Enfin, afin d'aborder le problème instationnaire, on montre la convergence du schéma de volumes finis pour une équation de convection–diffusion modèle avec second membre appartenant à L^1. Les estimations *a priori* de la solution et de sa dérivée en temps sont obtenues dans des normes discrètes dont les espaces correspondants ne sont pas duaux. Un résultat de compacité plus général que le théorème de Kolmogorov usuel, qui se pose comme un équivalent discret du Lemme d'Aubin–Simon, est alors proposé et permet de conclure à la convergence dans L^1 d'une suite de solutions discrètes.

Numerical schemes for one-point closure turbulence models

Summary: First-order Reynolds Averaged Navier–Stokes (RANS) turbulence models are studied in this thesis. These latter consist of the Navier–Stokes equations, supplemented with a system of balance equations describing the evolution of characteristic scalar quantities called "turbulent scales". In so doing, the contribution of the turbulent agitation to the momentum can be determined by adding a diffusive coefficient (called "turbulent viscosity") in the Navier-Stokes equations, such that it is defined as a function of the turbulent scales. The numerical analysis problems, which are studied in this dissertation, are treated in the frame of a fractional step algorithm, consisting of an approximation on regular meshes of the Navier–Stokes equations by the nonconforming Crouzeix–Raviart finite elements, and a set of scalar convection–diffusion balance equations discretized by the standard finite volume method.
A monotone numerical scheme based on the standard finite volume method is proposed so as to ensure that the turbulent scales, like the turbulent kinetic energy (k) and its dissipation rate (ε), remain positive in the case of the standard $k - \varepsilon$ model, as well as the $k - \varepsilon$ *RNG* and the extended $k - \varepsilon - \bar{v}^2 - f$ models.
The convergence of the proposed numerical scheme is then studied on a system composed of the incompressible Stokes equations and a steady convection–diffusion equation, which are both coupled by the viscosities and the turbulent production term. This reduced model allows to deal with the main difficulty encountered in the analysis of such problems: the definition of the turbulent production term leads to consider a class of convection–diffusion problems with an irregular right-hand side belonging to L^1.
Finally, to step towards the unsteady problem, the convergence of the finite volume scheme for a model convection–diffusion equation with L^1 data is proved. The *a priori* estimates on the solution and on is time derivative are obtained in discrete norms, for which corresponding continuous spaces are not dual. Consequently a more general compactness result than the Kolmogorov theorem is proved, which can be seen as a discrete counterpart of the Aubin–Simon lemma. This result allows to conclude to the convergence in L^1 of a sequence of discrete functions to a solution of the continuous problem.

Mots-clefs : Modèles de turbulence, éléments finis de Crouzeix–Raviart, schémas volumes finis, problème à données L^1.

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