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SCHÉMAS NUMÉRIQUES EXPLICITES POUR LE CALCUL D'ÉCOULEMENTS COMPRESSIBLES

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This manuscript is organized as follows. The first chapter is a general synthesis of the whole work; conclusion and perspectives are given at the end of this chapter. Then, the following three chapters provide a detailed exposition of the results stated in the synthesis.

Chapter 1

General synthesis

1.1 Introduction

The objective pursued in this work is to develop and study, from a theoretical point of view, an explicit scheme for the simulation of non viscous compressible flows, modelled either by the barotropic Euler equations or by the full Euler equations for an ideal gas. Our basic choice is to use an explicit variant of implicit and semi-implicit schemes that were developed and studied recently in the framework of the simulation of compressible flows at all speeds [17, 29, 26, 27]; in these latter works, the implicit scheme is studied as a first step in the mathematical analysis of pressure correction schemes, which extend algorithms that are classical in the incompressible framework; these are based on (*inf-sup* stable) staggered discretizations. In our approach, the upwinding techniques which are implemented for stability reasons are performed for each equation separately and with respect to the material velocity only. This is in contradiction with the most common strategy adopted for hyperbolic systems, where upwinding is built from the wave structure of the system (see *e.g.* [61, 6] for surveys). However, it yields algorithms which are used in practice (see *e.g.* the so-called AUSM family of schemes [45, 44]), because of their generality (a closed-form solution of Riemann problems is not needed), their implementation simplicity and their efficiency, thanks to an easy construction of the fluxes at the cell faces. But these schemes are scarcely studied from a theoretical point of view; one of our main concerns here will thus be to bring, as far as possible, theoretical arguments supporting our numerical developments.

We give in this chapter a review of the results obtained for the explicit version of the schemes in the case of the (inviscid) Euler equations, and refer to [28] for a review of the results of the implicit and semi–implicit versions, to [26, 27] for the detailed proofs of the results, and to [18] for the implementation of the pressure correction scheme in the case of a drift-diffusion model for two phase flows.

The chapter is organized as follows. We start by the description of the staggered meshes which are used for the discretization in space, using either a finite volume – non-conforming finite element or a full “MAC-type” finite volume scheme. We then study the scheme for the barotropic Euler equations in Section 1.3, for the full Euler equations in Section 1.4, and for the radial compressible flows in Section 1.5. Finaly, some numerical results are given in Section 1.6 to confort theoretical results.

1.2 Meshes and unknowns

Let $\mathcal{M}$ be a decomposition of the domain Ω, supposed to be regular in the usual sense of the finite element literature (*e.g.* [10]). The cells may be:

- for a general domain Ω, either convex quadrilaterals ($d = 2$) or hexahedra ($d = 3$) or simplices, both type of cells being possibly combined in a same mesh,
- for a domain the boundaries of which are hyperplanes normal to a coordinate axis, rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$) (the faces of which, of course, are then also necessarily normal to a coordinate axis).

By $\mathcal{E}$ and $\mathcal{E}(K)$ we denote the set of all $(d-1)$-faces σ of the mesh and of the element $K \in \mathcal{M}$ respectively. The set of faces included in the boundary of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal ones (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$; a face $\sigma \in \mathcal{E}_{\rm int}$ separating the cells K and L is denoted by $\sigma = K|L$. The outward normal vector to a face σ of K is denoted by $\boldsymbol{n}_{K,\sigma}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $|K|$ the measure of K and by $|\sigma|$ the $(d-1)$-measure of the face σ. For $1 \leq i \leq d$, we denote by $\mathcal{E}^{(i)} \subset \mathcal{E}$ and $\mathcal{E}_{\rm ext}^{(i)} \subset \mathcal{E}_{\rm ext}$ the subset of the faces of $\mathcal{E}$ and $\mathcal{E}_{\rm ext}$ respectively which are perpendicular to the i^{th} unit vector of the canonical basis of $\mathbb{R}^d$.

The space discretization is staggered, using either the Marker-And Cell (MAC) scheme [25, 24], or nonconforming low-order finite element approximations, namely the Rannacher and Turek element (RT) [57] for quadrilateral or hexahedric meshes, or the lowest degree Crouzeix-Raviart (CR) element [11] for simplicial meshes.

For all these space discretizations, the degrees of freedom for the pressure, the density and the internal energy (*i.e.* the discrete pressure, density and internal energy unknowns) are associated to the cells of the mesh $\mathcal{M}$, and are denoted by:

$$\big\{p_K,\ \rho_K,\ e_K,\ K \in \mathcal{M}\big\}.$$

Let us then turn to the degrees of freedom for the velocity (*i.e.* the discrete velocity unknowns).

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – The degrees of freedom for the velocity components are located at the center of the faces of the mesh, and we choose the version of the element where they represent the average of the velocity through a face. The set of degrees of freedom reads:

$$\{u_{\sigma,i},\ \sigma \in \mathcal{E},\ 1 \leq i \leq d\}.$$

- **MAC** discretization – The degrees of freedom for the i^{th} component of the velocity are defined at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so the whole set of discrete velocity unknowns reads:

$$\big\{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \leq i \leq d\big\}.$$

We now introduce a dual mesh, which will be used for the finite volume approximation of the time derivative and convection terms in the momentum balance equation.

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – For the RT or CR discretizations, the dual mesh is the same for all the velocity components. When $K \in \mathcal{M}$ is a simplex, a rectangle or a cuboid, for $\sigma \in \mathcal{E}(K)$, we define $D_{K,\sigma}$ as the cone with basis σ and with vertex the mass center of K (see Figure 1.1). We thus obtain a partition of K in m sub-volumes, where m is the number of faces of the mesh, each sub-volume having the same measure $|D_{K,\sigma}| = |K|/m$. We extend this definition to general quadrangles and hexahedra, by supposing that we have built a partition still of equal-volume sub-cells, and with the same connectivities; note that this is of course always possible, but that such a volume $D_{K,\sigma}$ may be no longer a cone; indeed, if K is far from a parallelogram, it may not be possible to build a cone having σ as basis, the opposite vertex lying in K and a volume equal to $|K|/m$. The volume $D_{K,\sigma}$ is referred to as the half-diamond cell associated to

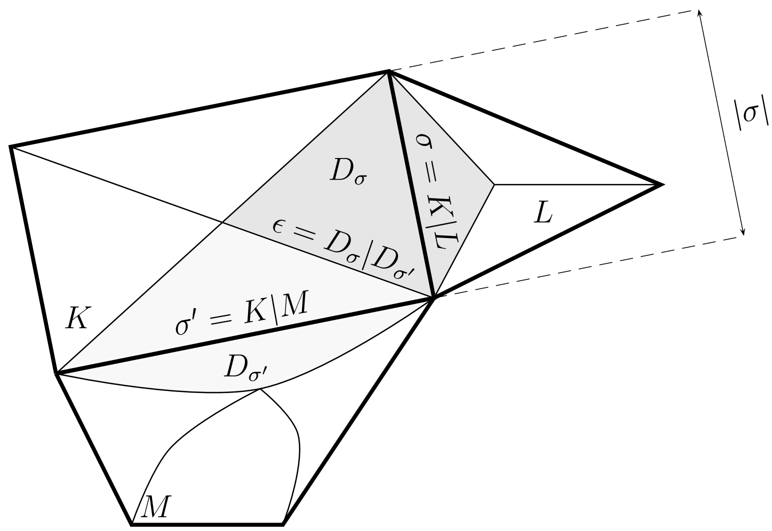


Figure 1.1: Primal and dual meshes for the Rannacher-Turek and Crouzeix-Raviart elements.

K and σ.

For $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we now define the diamond cell D_σ associated to σ by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$; for an external face $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)$, D_σ is just the same volume as $D_{K,\sigma}$.

- **MAC** discretization – For the MAC scheme, the dual mesh depends on the component of the velocity. For each component, the MAC dual mesh only differs from the RT or CR dual mesh by the choice of the half-diamond cell, which, for $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, is now the rectangle or rectangular parallelepiped of basis σ and of measure $|D_{K,\sigma}| = |K|/2$.

We denote by $|D_\sigma|$ the measure of the dual cell D_σ, and by $\epsilon = D_\sigma|D_{\sigma'}$ the face separating two diamond cells D_σ and $D_{\sigma'}$.

Finally, we need to deal with the impermeability (*i.e.* $\boldsymbol{u} \cdot \boldsymbol{n} = 0$) boundary condition. Since the velocity unknowns lie on the boundary (and not inside the cells), these conditions are taken into account in the definition of the discrete spaces. To avoid technicalities in the expression of the schemes, we suppose throughout this thesis that the boundary is *a.e.* normal to a coordinate axis, (even in the case of the RT or CR discretizations), which allows to simply set to zero the corresponding velocity unknowns:

$$\text{for } i = 1, \ldots, d,\ \forall \sigma \in \mathcal{E}_{\rm ext}^{(i)}, \qquad u_{\sigma,i} = 0. \tag{1.1}$$

Therefore, there are no degrees of freedom for the velocity on the boundary for the MAC scheme, and there are only $d-1$ degrees of freedom on each boundary face for the CR and RT discretizations, which depend on the orientation of the face. In order to be able to write a unique expression of the discrete equations for both MAC and CR/RT schemes, we introduce the set of faces $\mathcal{E}_{\mathcal{S}}^{(i)}$ associated to the degrees of freedom of each component of the velocity ($\mathcal{S}$ stands for "scheme"):

$$\mathcal{E}_{\mathcal{S}}^{(i)} = \left| \begin{array}{ll} \mathcal{E}^{(i)} \setminus \mathcal{E}_{\rm ext}^{(i)} & \text{for the MAC scheme}, \\ \mathcal{E} \setminus \mathcal{E}_{\rm ext}^{(i)} & \text{for the CR or RT scheme.} \end{array} \right.$$

Similarly, we unify the notation for the set of dual faces for both schemes by defining:

$$\tilde{\mathcal{E}}_{\mathcal{S}}^{(i)} = \left| \begin{array}{ll} \tilde{\mathcal{E}}^{(i)} \setminus \tilde{\mathcal{E}}_{\rm ext}^{(i)} & \text{for the MAC scheme}, \\ \tilde{\mathcal{E}} \setminus \tilde{\mathcal{E}}_{\rm ext}^{(i)} & \text{for the CR or RT scheme,} \end{array} \right.$$

where the symbol ˜ refers to the dual mesh; for instance, $\tilde{\mathcal{E}}^{(i)}$ is thus the set of faces of the dual mesh associated to the i^{th} component of the velocity, and $\tilde{\mathcal{E}}_{\rm ext}^{(i)}$ stands for the subset of these dual faces included in the boundary. Note that, for the MAC scheme, the faces of $\tilde{\mathcal{E}}^{(i)}$ are perpendicular to a unit vector of the canonical basis of $\mathbb{R}^d$, but not necessarily to the i^{th} one.

Note that general domains can easily be addressed (of course, with the CR or RT discretizations) by redefining, through linear combinations, the degrees of freedom at the external faces, so as to introduce the normal velocity as a new degree of freedom.

1.3 The compressible barotropic Euler equations

We address in this section the so-called barotropic Euler equations, which consist in the following system of partial differential equations:

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{1.2a}$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0, \tag{1.2b}$$

$$p = \wp(\rho) = \rho^\gamma. \tag{1.2c}$$

This problem is posed over an open bounded connected subset Ω of $\mathbb{R}^d$, $1 \leq d \leq 3$, of boundary $\partial\Omega$, and a finite time interval $(0, T)$. The variable t stands for the time, ρ, $\boldsymbol{u} = (u_1, \ldots, u_d)$ and p are the density, velocity and pressure in the flow. The three above equations are respectively the mass balance, the momentum balance and the equation of state of the fluid, which is supposed to take the form $\wp(s) = s^\gamma$, where $\gamma \geq 1$ is a coefficient which is specific to the fluid considered. This system must be supplemented by initial conditions for ρ and $\boldsymbol{u}$, denoted by ρ_0 and $\boldsymbol{u}_0$, and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition, which we suppose to be:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0,$$

at any time and *a.e.* on $\partial\Omega$, where $\boldsymbol{n}$ stands for the normal vector to the boundary.

Let us denote by E_k the kinetic energy $E_k = \frac{1}{2}|\boldsymbol{u}|^2$. Taking the inner product of (1.2b) by $\boldsymbol{u}$ yields, after formal compositions of partial derivatives and using (1.2a):

$$\partial_t(\rho E_k) + \operatorname{div}\big(\rho\, E_k\, \boldsymbol{u}\big) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0. \tag{1.3}$$

This relation is referred to as the kinetic energy balance.

Let us now define the function $\mathcal{P}$, from $(0, +\infty)$ to $\mathbb{R}$, as a primitive of $s \mapsto \wp(s)/s^2$; this quantity is often called the elastic potential. Let $\mathcal{H}$ be the function defined by $\mathcal{H}(s) = s\mathcal{P}(s),\ \forall s \in (0, +\infty)$, which, for the specific equation of state used here, yields:

$$\mathcal{H}(s) = s\,\mathcal{P}(s) = \begin{cases} \dfrac{s^\gamma}{\gamma - 1} & \text{if } \gamma > 1, \\ s \ln(s) & \text{if } \gamma = 1. \end{cases} \tag{1.4}$$

Since $\wp$ is an increasing function, $\mathcal{H}$ is convex. In addition, it may easily be checked that $\rho\mathcal{H}'(\rho) - \mathcal{H}(\rho) = \wp(\rho)$. Therefore, by a formal computation, detailed for instance in [26, Appendix], multiplying (1.2a) by $\mathcal{H}'(\rho)$ yields:

$$\partial_t\big(\mathcal{H}(\rho)\big) + \operatorname{div}\big(\mathcal{H}(\rho)\,\boldsymbol{u}\big) + p\operatorname{div}(\boldsymbol{u}) = 0. \tag{1.5}$$

Let us denote by $\mathcal{S}$ the quantity $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$. Summing (1.3) and (1.5), we get:

$$\partial_t\mathcal{S} + \operatorname{div}\big((\mathcal{S}+p)\,\boldsymbol{u}\big) = 0. \tag{1.6}$$

In fact, to avoid invoking unrealistic regularity assumptions, such a computation should be done on the regularized equations (obtained by adding diffusion perturbation terms), and, when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (1.6), so that we get in the distribution sense:

$$\partial_t\mathcal{S} + \operatorname{div}\big((\mathcal{S}+p)\,\boldsymbol{u}\big) \leq 0. \tag{1.7}$$

The quantity $\mathcal{S}$ is an entropy of the system, and an entropy solution to (1.2) is thus required to satisfy:

$$\begin{aligned}
&\forall\varphi \in \mathrm{C}_c^\infty\big(\Omega\times[0,T)\big),\ \varphi \geq 0,\\
&\qquad \int_0^T\int_\Omega \big[-\mathcal{S}\partial_t\varphi - (\mathcal{S}+p)\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \mathcal{S}_0\,\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} \leq 0,
\end{aligned} \tag{1.8}$$

with $\mathcal{S}_0 = \frac{1}{2}\rho_0|\boldsymbol{u}_0|^2 + \mathcal{H}(\rho_0)$. Then, since the normal velocity is prescribed to zero at the boundary, integrating (1.7) over Ω yields:

$$\frac{d}{dt}\int_\Omega \big[\frac{1}{2}\,\rho\,|\boldsymbol{u}|^2 + \mathcal{H}(\rho)\big]\,\mathrm{d}\boldsymbol{x} \leq 0. \tag{1.9}$$

Since $\rho \geq 0$ by (1.2a) (and the associated initial and boundary conditions) and the function $s \mapsto \mathcal{H}(s)$ is bounded by below and increasing at least for s large enough, Inequality (1.9) provides an estimate on the solution.

The purpose of this section is to build an explicit scheme for the numerical solution of System (1.2), and prove the following results:

- a discrete kinetic energy balance (*i.e.* a discrete analogue of (1.3)) is established

on dual cells, while a discrete potential elastic balance (*i.e.* a discrete analogue of (1.5)) is established on primal cells.
Note however that, because of residual terms appearing in the potential elastic balance, contrary to what is obtained for implicit and semi-implicit variants of the present scheme [15, 26], these equations do not seem to yield the stability of the scheme (*i.e.* a discrete global entropy conservation analogue to Equation (1.9)), at least without supposing drastic limitations of the time step.

- In one space dimension, the limit of any convergent sequence of solutions to the scheme is shown to be a weak solution to the continuous problem, and thus to satisfy the Rankine-Hugoniot conditions.
- Finally, passing to the limit in the discrete kinetic energy and elastic potential balances, such a limit is also shown to satisfy the entropy inequality (1.8), see Theorem 1.3.6 below.

1.3.1 The scheme

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n = 0, \tag{1.10a}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma, \tag{1.10b}$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$,

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n u_{\epsilon,i}^n + |D_\sigma| \, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0, \tag{1.10c}$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (1.10a) is obtained by the discretization of the mass balance equation (1.2a) over the primal mesh, and $F_{K,\sigma}^{n+1}$ stands for the mass flux across σ outward K, which, because of the impermeability condition, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad F^n_{K,\sigma} = |\sigma|\ \rho^n_\sigma\ u^n_{K,\sigma}, \tag{1.11}$$

where $u^n_{K,\sigma}$ is an approximation of the normal velocity to the face σ outward K. This latter quantity is defined by:

$$u^n_{K,\sigma} = \left| \begin{array}{l} u^n_{\sigma,i}\ \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\[2ex] \boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{K,\sigma} \text{ in the CR and RT cases,} \end{array} \right. \tag{1.12}$$

where $\boldsymbol{e}^{(i)}$ denotes the i-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face $\sigma = K|L$ is approximated by the upwind technique:

$$\rho^n_\sigma = \left| \begin{array}{ll} \rho^n_K & \quad \text{if } u^n_{K,\sigma} \geq 0, \\[2ex] \rho^n_L & \quad \text{otherwise.} \end{array} \right. \tag{1.13}$$

We now turn to the discrete momentum balance (1.10c), which is obtained by discretizing the momentum balance equation (1.2b) on the dual cells associated to the faces of the mesh. For the discretization of the time derivative, we must provide a definition for the values $\rho^{n+1}_{D_\sigma}$ and $\rho^n_{D_\sigma}$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\rm int},\ \text{for } k = n \text{ and } k = n+1, \qquad |D_\sigma|\ \rho^k_{D_\sigma} = |D_{K,\sigma}|\ \rho^k_K + |D_{L,\sigma}|\ \rho^k_L. \tag{1.14}$$

Let us then turn to the discretization of the convection term. The first task is to define the discrete mass flux through the dual face ϵ outward D_σ, denoted by $F^n_{\sigma,\epsilon}$; the guideline for its construction is that a finite volume discretization of the mass balance equation over the diamond cells, of the form

$$\forall \sigma \in \mathcal{E}, \qquad \frac{|D_\sigma|}{\delta t}\,(\rho^{n+1}_{D_\sigma} - \rho^n_{D_\sigma}) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F^n_{\sigma,\epsilon} = 0, \tag{1.15}$$

must hold in order to be able to derive a discrete kinetic energy balance (see Section 1.3.2 below). For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [30]. In the case of the CR and RT schemes, for a dual face ϵ included in the primal cell K, this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the face and the cell) of the mass fluxes through the faces of K, *i.e.* the quantities $(F^{n+1}_{K,\sigma})_{\sigma\in\mathcal{E}(K)}$ appearing in the discrete mass balance (1.10a). We refer to [1, 17] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across that face is zero. Therefore, the values $u^{n+1}_{\epsilon,i}$ are only needed at the internal dual faces, and we make the upwind choice for their discretization:

$$\text{for } \epsilon = D_\sigma | D'_\sigma, \quad u^n_{\epsilon,i} = \left| \begin{array}{ll} u^n_{\sigma,i} & \text{if } F^n_{\sigma,\epsilon} \geq 0, \\ u^n_{\sigma',i} & \text{otherwise.} \end{array} \right. \tag{1.16}$$

The last term $(\boldsymbol{\nabla} p)^{n+1}_{\sigma,i}$ stands for the i-th component of the discrete pressure gradient at the face σ. The gradient operator is built as the transpose of the discrete operator for the divergence of the velocity, the discretization of which is based on the primal mesh. Let us denote the divergence of $\boldsymbol{u}^{n+1}$ over $K \in \mathcal{M}$ by $(\mathrm{div}\boldsymbol{u})^{n+1}_K$; its natural approximation reads:

$$\text{for } K \in \mathcal{M}, \qquad (\mathrm{div}\boldsymbol{u})^{n+1}_K = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \; u^{n+1}_{K,\sigma}. \tag{1.17}$$

Consequently, the components of the pressure gradient are given by:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i} = \frac{|\sigma|}{|D_\sigma|} \, (p^{n+1}_L - p^{n+1}_K) \; \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}, \tag{1.18}$$

this expression being derived thanks to the following duality relation with respect to the L^2 inner product:

$$\sum_{K\in\mathcal{M}} |K| \; p^{n+1}_K \; (\mathrm{div}\boldsymbol{u})^{n+1}_K + \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}_{\mathcal{S}}} |D_\sigma| \; u^{n+1}_{\sigma,i} \; (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i} = 0. \tag{1.19}$$

Note that, because of the impermeability boundary conditions, the discrete gradient is not defined at the external faces.

Finally, the initial approximations for ρ and $\boldsymbol{u}$ are given by the average of the initial conditions ρ_0 and $\boldsymbol{u}_0$ on the primal and dual cells respectively:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \\ &\text{for } 1 \leq i \leq d, \ \forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \quad u_{\sigma,i}^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} (\boldsymbol{u}_0(\boldsymbol{x}))_i \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{1.20}$$

Note that, thanks to the upwind choice in the mass balance equation (1.10a), if ρ^n is positive in (1.10a), then so is ρ^{n+1} under the following CFL condition:

$$\delta t \leq \frac{|K|}{\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, (u_{K,\sigma}^n)^+}, \tag{1.21}$$

where, for $a \in \mathbb{R}$, $a^+ = \max(a, 0)$. Since, by assumption, ρ_0 is positive, under Condition (1.21), the discrete density thus remains positive at all times.

1.3.2 Kinetic energy balance and elastic potential identity

We begin by deriving a discrete kinetic energy balance equation, as was already done for the implicit and fractional time step scheme described in [26]. We follow the same lines as in the classical derivation of the kinetic energy balance equation (1.3) in the continuous setting: the discrete kinetic energy balance is obtained by multiplying the (i^{th} component of the) momentum balance equation (1.10c) associated to the face σ by $u_{\sigma,i}^{n+1}$, summing over the components and using the mass balance equation (1.10a) twice.

Lemma 1.3.1 (Discrete kinetic energy balance)**.**
A solution to the system (1.10) *satisfies the following equality, for* $1 \leq i \leq d$*,* $\sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$ *and* $0 \leq n \leq N-1$*:*

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \Big[\rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \Big] + \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n \, (u_{\epsilon,i}^n)^2 + |D_\sigma| \, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} \, u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \tag{1.22}$$

with:

$$R_{\sigma,i}^{n+1} = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\rho_{D_\sigma}^{n+1}(u_{\sigma,i}^{n+1} - u_{\sigma,i}^{n})^2 + \frac{1}{2}\sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F_{D_\sigma,\epsilon}^n)^-(u_{\sigma',i}^n - u_{\sigma,i}^n)^2 - \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F_{D_\sigma,\epsilon}^n)^-(u_{\sigma',i}^n - u_{\sigma,i}^n)\,(u_{\sigma,i}^{n+1} - u_{\sigma,i}^n), \quad (1.23)$$

where, for $a \in \mathbb{R}$*,* $a^- \geq 0$ *is defined by* $a^- = -\min(a,0)$.

Equation (1.22) is a discrete analogue of Equation (1.3), with an upwind discretization of the convection term. The remainder term $R_{\sigma,i}^{n+1}$ is non-negative under the following CFL condition:

$$\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \qquad \delta t \leq \frac{|D_\sigma|\ \rho_{D_\sigma}^{n+1}}{\displaystyle\sum_{\epsilon\in\mathcal{E}(D_\sigma)} (F_{\sigma,\epsilon}^n)^-}. \quad (1.24)$$

Similarly, the solution to the scheme (1.10) satisfies a discrete version of the elastic potential identity (1.5), which we now state.

Lemma 1.3.2 (Discrete potential balance)**.** *Let* $\mathcal{H}$ *be defined by* (1.4)*. A solution to the system* (1.10) *satisfies the following equality, for* $K \in \mathcal{M}$ *and* $0 \leq n \leq N-1$*:*

$$\frac{|K|}{\delta t}\Big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\Big] + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ \mathcal{H}(\rho_\sigma^n)\ u_{K,\sigma}^n + |K|\ p_K^n(\mathrm{div}\boldsymbol{u}^n)_K = -R_K^{n+1}, \quad (1.25)$$

with:

$$R_K^{n+1} = \frac{1}{2}\frac{|K|}{\delta t}\ \mathcal{H}''(\overline{\rho}_{K,1}^n)\ (\rho_K^{n+1} - \rho_K^n)^2 + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ u_{K,\sigma}^n\ \mathcal{H}''(\overline{\rho}_{K,2}^n)\ \rho_\sigma^n\ (\rho_K^{n+1} - \rho_K^n) + \frac{1}{2}\sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\ (u_{K,\sigma}^n)^-\ \mathcal{H}''(\overline{\rho}_\sigma^n)\ (\rho_K^n - \rho_L^n)^2, \quad (1.26)$$

where $\overline{\rho}_{K,1}^n,\ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$*, and* $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_L^n]\!]$ *for all* $\sigma \in \mathcal{E}(K)$*, where, for* $a,\ b \in \mathbb{R}$*, we denote by* $[\![a,b]\!]$ *the interval* $[\![a,b]\!] = \{\theta a + (1-\theta)b,\ \theta \in [0,1]\}$.

Unfortunately, it does not seem that $R_K^{n+1} \geq 0$ in any case, and so we are not able to prove a discrete counterpart of the total entropy estimate (1.9), which would yield a stability estimate for the scheme. However, under a condition for a time step which is only slightly more restrictive than a CFL-condition, and under some stability assumptions for the solutions to the scheme, we are able to show that this remainder term tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$, which allows to conclude, in the 1D case, that a convergent sequence of solutions satisfies the entropy inequality (1.8): this is the result stated in Theorem 1.3.6 below.

1.3.3 Passing to the limit in the scheme

The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

The 1D version of the scheme which is studied in this section may be obtained from Scheme (1.10) by taking the MAC variant of the scheme, using only one horizontal stripe of grid cells, supposing that the vertical component of the velocity (the degrees of freedom of which are located on the top and bottom boundaries) vanishes, and that the measure of the vertical faces is equal to 1. For the sake of readability, however, we completely rewrite this 1D scheme, and, to this purpose, we first introduce some adaptations of the notations to the one dimensional case. For any $K \in \mathcal{M}$, we denote by h_K its length (so $h_K = |K|$); when we write $K = [\sigma\sigma']$, this means that either $K = (x_\sigma, x_{\sigma'})$ or $K = (x_{\sigma'}, x_\sigma)$; if we need to specify the order, *i.e.* $K = (x_\sigma, x_{\sigma'})$ with $x_\sigma < x_{\sigma'}$, then we write $K = [\overrightarrow{\sigma\sigma'}]$. For an interface $\sigma = K|L$ between two cells K and L, we define $h_\sigma = (h_K + h_L)/2$, so, by definition of the dual mesh, $h_\sigma = |D_\sigma|$. If we need to specify the order of the cells K and L, say K is left of L, then we write $\sigma = \overrightarrow{K|L}$. With these notations, the explicit scheme (1.10) may be written as follows in the one dimensional setting:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(x)\,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \tag{1.27a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$

$$\frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{1.27b}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{1.27c}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int},$$
$$\frac{|D_\sigma|}{\delta t}(\rho^{n+1}_{D_\sigma} u^{n+1}_\sigma - \rho^n_{D_\sigma} u^n_\sigma) + F^n_L u^n_L - F^n_K u^n_K + p^{n+1}_L - p^{n+1}_K = 0, \tag{1.27d}$$

The mass flux in the discrete mass balance equation is given, for $\sigma \in \mathcal{E}_{\rm int}$, by $F^n_\sigma = \rho^n_\sigma u^n_\sigma$, where the upwind approximation for the density at the face, ρ^n_σ, is defined by (1.13). In the momentum balance equation, the application of the procedure described in Section 1.3.1 yields for the density associated to the dual cell D_σ with $\sigma = K|L$ and for the mass fluxes at the dual face located at the center of the mesh $K = [\overrightarrow{\sigma\sigma'}]$:

$$\text{for } k = n \text{ and } k = n+1,\ \rho^k_{D_\sigma} = \frac{1}{2\,|D_\sigma|}\,(|K|\,\rho^k_K + |L|\,\rho^k_L), \qquad F^n_K = \frac{1}{2}\,(F^n_\sigma + F^n_{\sigma'}), \tag{1.28}$$

and the approximation of the velocity at this face is upwind: $u^n_K = u^n_\sigma$ if $F^n_K \geq 0$ and $u^n_K = u^n_{\sigma'}$ otherwise.

Definition 1.3.3 (Regular sequence of discretizations)**.**
We define a regular sequence of discretizations $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ *as a sequence of meshes, time steps and numerical diffusion coefficients satisfying:*

(i) *both the time step* $\delta t^{(m)}$ *and the size* $h^{(m)}$ *of the mesh* $\mathcal{M}^{(m)}$*, defined by* $h^{(m)} =$ *sup*$_{K\in\mathcal{M}^{(m)}}\ h_K$*, tend to zero as* $m \to \infty$*,*

(ii) *there exists* $\theta > 0$ *such that:*

$$\theta \leq \frac{h_K}{h_L} \leq \frac{1}{\theta}, \qquad \forall m \in \mathbb{N} \textit{ and } K,\ L \in \mathcal{M}^{(m)} \textit{ sharing a face,}$$

Let such a regular sequence of discretizations be given, and $\rho^{(m)}$, $p^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (1.27) with the mesh $\mathcal{M}^{(m)}$ and the time step $\delta t^{(m)}$. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho^{(m)}$, the pressure $p^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0,T)$ by:

$$\rho^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (\rho^{(m)})_K^n \, \mathcal{X}_K(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{1.29a}$$

$$u^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} (u^{(m)})_\sigma^n \, \mathcal{X}_{D_\sigma}(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{1.29b}$$

$$p^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (p^{(m)})_K^n \, \mathcal{X}_K(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{1.29c}$$

where $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{(n,n+1]}$ stand for the characteristic function of the intervals K, D_σ and $(t^n, t^{n+1}]$ respectively.

For discrete functions q and v defined on the primal and dual mesh, respectively, we define a discrete $\mathrm{L}^1((0,T); \mathrm{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\mathrm{int}}} |q_L^n - q_K^n|, \qquad \|v\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_{\mathrm{int}}} |v_{\sigma'}^n - v_\sigma^n|,$$

and a discrete $\mathrm{L}^1(\Omega; \mathrm{BV}((0,T)))$ norm by:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K \in \mathcal{M}} |K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma \in \mathcal{E}} |D_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

For the consistency result that we are seeking (Theorem 1.3.4 below), we have to assume that a sequence of discrete solutions $\left(\rho^{(m)}, p^{(m)}, u^{(m)}\right)_{m \in \mathbb{N}}$ satisfies $\rho^{(m)} > 0$ and $p^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (1.21) is satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T) \times \Omega)^3$, *i.e.*:

$$0 < (\rho^{(m)})_K^n \leq C \text{ and } 0 < (p^{(m)})_K^n \leq C, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \ \forall m \in \mathbb{N}, \tag{1.30}$$

and:

$$|(u^{(m)})_\sigma^n| \leq C, \quad \forall \sigma \in \mathcal{E}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \ \forall m \in \mathbb{N}, \tag{1.31}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0, e_0 and u_0 belong to $\mathrm{L}^\infty(\Omega)$. We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}. \tag{1.32}$$

and:

$$\|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}. \tag{1.33}$$

We are not able to prove the estimates (1.30)–(1.33) for the solutions of the scheme; however, such inequalities are satisfied by the "interpolation" (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* with a limited sequence of meshes and time steps).

A weak solution to the continuous problem satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\,\partial_t\varphi + \rho\,u\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{1.34a}$$

$$-\int_0^T \int_\Omega \Big[\rho\,u\,\partial_t\varphi + (\rho\,u^2 + p)\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,u_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{1.34b}$$

$$p = \rho^\gamma. \tag{1.34c}$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks* (*i.e.* shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions). This is the result we are seeking and which we now state.

Theorem 1.3.4 (Consistency of the one-dimensional explicit scheme, barotropic case)**.** *Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)},\, \delta t^{(m)})_{m\in\mathbb{N}}$ be a regular sequence of discretizations in the sense of Definition* 1.3.3*, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (1.30)–(1.32) *and converges in $\mathrm{L}^p(\Omega\times(0,T))^3$, for $1 \leq p < \infty$, to $(\bar\rho,\, \bar p,\, \bar u) \in \mathrm{L}^\infty(\Omega\times(0,T))^3$. We suppose in addition that both sequences $(\rho^{(m)})_{m\in\mathbb{N}}$ and $(1/\rho^{(m)})_{m\in\mathbb{N}}$ are uniformly bounded in*

$\mathrm{L}^\infty(\Omega \times (0,T))$.

Then the limit $(\bar\rho,\ \bar p,\ \bar u)$ *satisfies the system* (1.34).

Main ideas of the proof – It is clear that with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The proof of this theorem is thus obtained by passing to the limit in the scheme, first for the mass balance equation and then for the momentum balance equation. This is performed by considering a smooth function φ over the domain Ω, defining its interpolate $\varphi_{\mathcal M}$ over the cells and its interpolate $\varphi_{\mathcal E}$ over the edges. Then one first multiplies the discrete mass balance equation (1.27b) by the value φ_K of the interpolate of φ on K, sum over the cells and time steps, and, introducing the discrete time and space derivatives of φ and noting that they tend uniformly to the continuous time and space derivatives of φ, pass to the limit on all terms to recover (1.34a). Similarly, one multiplies the discrete momentum equation (1.27d) by the value φ_σ of φ on σ, sum over the edges and time steps, and again pass to the limit on all terms to recover (1.34b). The details of this proof may be found in Chapter 2, Theorem 2.4.2. □

We now turn to the entropy condition (1.8). To this purpose, we need to introduce the following additional condition for a sequence of discretizations:

$$\lim_{m\to+\infty} \frac{\delta t^{(m)}}{\min_{K\in\mathcal M^{(m)}} h_K} = 0. \tag{1.35}$$

Note that this condition is slightly more restrictive that a standard CFL condition. It allows to bound the remainder term in the discrete elastic potential balance as stated in the following lemma.

Lemma 1.3.5. *Let* Ω *be an open bounded interval of* $\mathbb R$. *Let* $(\mathcal M^{(m)},\ \delta t^{(m)})_{m\in\mathbb N}$ *be a sequence of discretizations such that the time step* $\delta t^{(m)}$ *tends to zero as* $m\to\infty$*, and* $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb N}$ *be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (1.30)–(1.31). *In addition, we assume that* $(\rho^{(m)})_{m\in\mathbb N}$ *satisfies the following uniform BV estimate:*

$$\|\rho^{(m)}\|_{\mathcal T,t,\mathrm{BV}} \le C, \quad \forall m\in\mathbb N, \tag{1.36}$$

and, for $\gamma < 2$ only, is uniformly bounded by below, i.e. *that there exists $c > 0$ such that:*

$$c \leq (\rho^{(m)})_K^n, \quad \forall K \in \mathcal{M}^{(m)},\ \textit{for}\ 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{1.37}$$

Let us suppose that the CFL condition (1.35) *hold. Let $\mathcal{R}^{(m)}$ be defined by:*

$$\mathcal{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} (R_K^{n+1})^-,$$

with R_K^{n+1} given by (1.26)*. Then:*

$$\lim_{m \to +\infty} \mathcal{R}^{(m)} = 0.$$

Then we are in position to state the following consistency result.

Theorem 1.3.6 (Entropy consistency, barotropic case)**.** *Under the assumptions of Theorem* 1.3.4 *and Lemma* 1.3.5*, the limit $(\bar\rho,\ \bar p,\ \bar u)$ satisfies the entropy condition* (1.8)*.*

Main ideas of the proof – The proof of this theorem is again based on a passage to the limit in discrete equations, namely the discrete kinetic balance equation (1.22) and the elastic potential balance (1.25). This computation is very close to the proof of consistency of the scheme for the full Euler equations with the total energy balance, *i.e.* the proof of Theorem 1.4.1 below. We refer to Chapter 2, Theorem 2.4.4 for the details of this computation. □

1.4 The full Euler equations

Let us now turn to the full compressible Euler equations, which read:

$$\partial_t \rho + \operatorname{div}(\rho\, \boldsymbol{u}) = 0, \tag{1.38a}$$

$$\partial_t(\rho\, \boldsymbol{u}) + \operatorname{div}(\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \tag{1.38b}$$

$$\partial_t(\rho\, E) + \operatorname{div}(\rho\, E\, \boldsymbol{u}) + \operatorname{div}(p\, \boldsymbol{u}) = 0, \tag{1.38c}$$

$$p = (\gamma - 1)\, \rho\, e, \qquad E = \frac{1}{2}|\boldsymbol{u}|^2 + e, \tag{1.38d}$$

where t stands for the time, ρ, $\boldsymbol{u}$, p, E and e are the density, velocity, pressure, total energy and internal energy respectively, and $\gamma > 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0, T)$, where Ω is an open bounded connected subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, and $(0, T)$ is a finite time interval.

System (1.38) is complemented by initial conditions for ρ, e and $\boldsymbol{u}$, denoted by ρ_0, e_0 and $\boldsymbol{u}_0$ respectively, with $\rho_0 > 0$ and $e_0 > 0$, and by a boundary condition which we suppose to be $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at any time and *a.e.* on $\partial\Omega$, where $\boldsymbol{n}$ stands for the normal vector to the boundary.

Let us suppose that the solution is regular. Substracting the kinetic energy balance equation (1.3) from the total energy balance (1.38c), we obtain the internal energy balance equation:

$$\partial_t(\rho e) + \operatorname{div}(\rho e \boldsymbol{u}) + p \operatorname{div}(\boldsymbol{u}) = 0. \tag{1.39}$$

Since,

- thanks to the mass balance equation, the first two terms in the left-hand side of (1.39) may be recast as a transport operator: $\partial_t(\rho e) + \operatorname{div}(\rho e \boldsymbol{u}) = \rho\, [\partial_t e + \boldsymbol{u} \cdot \boldsymbol{\nabla} e]$,
- and, from the equation of state, the pressure vanishes when $e = 0$,

this equation implies, if $e \geq 0$ at $t = 0$ and with suitable boundary conditions, that e remains non-negative at all times.

We wish to build an explicit version of the staggered implicit and semi-implicit schemes that have already been studied for the Euler equations [27]. As already mentioned in [27], discretizing (1.39) instead of the total energy balance (1.38c) presents two advantages:

- first, it avoids the space discretization of the total energy, which is rather unnatural for staggered schemes since the degrees of freedom for the velocity and the scalar variables are not collocated,
- second, a suitable discretization of (1.39) may yield, "by construction" of the scheme, the positivity of the internal energy.

However, for solutions with shocks, Equation (1.39) is not equivalent to (1.38c); more precisely speaking, at the locations of shocks, positive measures should appear, at the right-hand side of Equation (1.39). Discretizing (1.39) instead of (1.38c) may thus yield a scheme which does not compute the correct weak discontinuous solutions; in particular, the numerical solutions may present (smeared) shocks which do not satisfy the

Rankine-Hugoniot conditions associated to (1.38c). The essential result of this section is to provide solutions to circumvent this problem. To this purpose, we closely mimic the above performed formal computation:

- we start from the discrete kinetic energy balance (1.22), and remark that the residual terms at the right-hand side do no tend to zero with the space and time steps (they are the discrete manifestations of the above mentioned measures),
- we thus compensate these residual terms by corrective terms in the internal energy balance.

We provide a theoretical justification of this process by showing that, in the 1D case, if the scheme is stable and converges to a limit (in a sense to be defined), this limit satisfies a weak form of (1.38c) which implies the correct Rankine-Hugoniot conditions.

1.4.1 The scheme

With the same notations as in Section 1.3.1, we consider an explicit-in-time numerical scheme for the discretization of the Euler equations, *i.e.* System (1.38). In its fully discrete form, this scheme reads, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n = 0, \tag{1.40a}$$

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n e_\sigma^n + |K|\, p_K^n\, (\operatorname{div} \boldsymbol{u})_K^n = S_K^n, \tag{1.40b}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}, \tag{1.40c}$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$,

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n u_{\epsilon,i}^n + |D_\sigma|\, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0. \tag{1.40d}$$

Equations (1.40a) and (1.40d) are the same as the discrete mass and momentum balance equations (1.10a) and (1.10c) of the barotropic model and were described in Section

1.3.1. Equation (1.40b) is an approximation of the internal energy balance over the primal cell K. The positivity of the convection operator is ensured if we use an upwinding technique for this term [42]:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \qquad e_\sigma^n = \begin{vmatrix} e_K^n & \text{if } F_{K,\sigma}^n \geq 0, \\ e_L^n & \text{otherwise.} \end{vmatrix}$$

The discrete divergence of the velocity, $(\mathrm{div}\boldsymbol{u})_K^n$, is defined by (1.17) and the discrete pressure gradient by (1.18), so that the discrete gradient and divergence operators are dual with respect to the L^2 inner product, as stated in (1.19). The right-hand side, S_K^n, is derived using consistency arguments in the next section.

Finally, the initial approximations for ρ and $\boldsymbol{u}$ are given by (1.20) and the initial condition for e is the following one:

$$\forall K \in \mathcal{M}, \qquad e_K^0 = \frac{1}{|K|} \int_K e_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{1.41}$$

Since, by assumption, $e_0 \geq 0$, the (discrete) initial condition for the internal energy is positive. Using standard argument, thanks to the fact that, in the third term of (1.40b), p_K^n is proportional to e_K^n (precisely speaking, $p_K^n = (\gamma - 1)\, \rho_K^n \, e_K^n$), we prove that the internal energy remains positive at all times assuming (1.21) and the following additional CFL condition:

$$\delta t \leq \frac{|K|}{\gamma \displaystyle\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, (u_{K,\sigma}^n)^+}, \quad \forall K \in \mathcal{M}, \tag{1.42}$$

provided that S_K^n is positive, which is the case under another CFL condition (see Inequality (1.45) in the next section).

1.4.2 The discrete kinetic energy balance equation and the corrective source terms

By Lemma 1.3.1, we know that a discrete kinetic balance holds, with, at the right-hand side, some residual terms. The next step is now to define corrective terms in the internal energy balance, with the aim to recover a consistent discretization of the total energy balance. The first idea to do this could be just to sum the (discrete) kinetic energy balance with the internal energy balance: it is indeed possible for a collocated discretization. But here, we face the fact that the kinetic energy balance is associated to the dual mesh, while the internal energy balance is discretized on the primal one. The way to circumvent this difficulty is to remark that we do not really need a discrete total energy balance; in fact, we only need to recover (a weak form of) this equation when the mesh and time steps tend to zero. To this purpose, we choose the quantities (S_K^n) in such a way as to somewhat compensate the terms $(R_{\sigma,i}^n)$ given by (1.23):

$$\forall K \in \mathcal{M},\ S_K^{n+1} = \sum_{i=1}^{d} S_{K,i}^{n+1} \text{ with } \qquad S_{K,i}^{n+1} = \frac{1}{2}\,\rho_K^{n+1} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathcal{S}}^{(i)}} \frac{|D_{K,\sigma}|}{\delta t}\left(u_{\sigma,i}^{n+1} - u_{\sigma,i}^{n}\right)^2$$
$$+ \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}_{\mathcal{S}}^{(i)},\ \epsilon\cap\bar{K}\neq\emptyset,\\ \epsilon=D_\sigma|D_{\sigma'},\, F_{\sigma,\epsilon}^n\leq 0}} \alpha_{K,\epsilon}\left[\frac{|F_{\sigma,\epsilon}^n|}{2}\,(u_{\sigma,i}^n - u_{\sigma',i}^n)^2 + F_{\sigma,\epsilon}^n\left(u_{\sigma,i}^{n+1} - u_{\sigma,i}^n\right)\,(u_{\sigma,i}^n - u_{\sigma',i}^n)\right].$$

The coefficient $\alpha_{K,\epsilon}$ is fixed to 1 if the face ϵ is included in K, and this is the only situation to consider for the RT and CR discretizations. For the MAC scheme, some dual faces are included in the primal cells, but some lie on their boundary; for such a boundary edge ϵ, we denote by $\mathcal{N}_\epsilon$ the set of cells M such that $\bar{M}\cap\epsilon\neq\emptyset$ (the cardinal of this set is always 4), and compute $\alpha_{K,\epsilon}$ by:

$$\alpha_{K,\epsilon} = \frac{|K|}{\sum_{M\in\mathcal{N}_\epsilon}|M|}. \tag{1.43}$$

For a uniform grid, this formula yields $\alpha_{K,\epsilon} = 1/4$.

The expression of the terms $(S_K^{n+1})_{K\in\mathcal{M}}$ is justified by the passage to the limit in the scheme (for a one-dimensional problem) performed in Section 1.4.3. We can first note that:

$$\sum_{K\in\mathcal{M}} S_K^{n+1} - \sum_{i=1}^{d}\sum_{\sigma\in\mathcal{E}_{\mathcal{S}}^{(i)}} R_{\sigma,i}^{n+1} = 0. \tag{1.44}$$

Indeed, the first part of $S_{K,i}^{n+1}$, thanks to the expression (1.28) of the density at the face $\rho_{D_\sigma}^{n+1}$, results from a dispatching of the first part of the residual over the two adjacent cells:

$$\frac{1}{2}\,\frac{|D_\sigma|}{\delta t}\,\rho^{n+1}_{D_\sigma}\,\left(u^{n+1}_{\sigma,i}-u^{n}_{\sigma,i}\right)^2 = \underbrace{\frac{1}{2}\,\frac{|D_{K,\sigma}|}{\delta t}\,\rho^{n+1}_{K}\,\left(u^{n+1}_{\sigma,i}-u^{n}_{\sigma,i}\right)^2}_{\text{affected to K}} + \underbrace{\frac{1}{2}\,\frac{|D_{L,\sigma}|}{\delta t}\,\rho^{n+1}_{L}\,\left(u^{n+1}_{\sigma,i}-u^{n}_{\sigma,i}\right)^2}_{\text{affected to L}}.$$

The same argument holds for the terms associated to the dual faces, which explains, in particular, the definition of the coefficients $\alpha_{K,\epsilon}$. The scheme thus conserves the discrete equivalent of the integral of the total energy over the computational domain.

Using Young's inequality and remarking that, for all the considered discretizations,

$$\sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathcal{S}}^{(i)}} |D_{K,\sigma}| = |K|,$$

we obtain that $S^{n+1}_{K,i}$ in non-negative provided that the following CFL condition holds:

$$\delta t \leq \frac{|D_{K,\sigma}|\;\rho^{n+1}_{K}}{\displaystyle\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\ \epsilon\cap\bar{K}\neq\emptyset} \alpha_{K,\epsilon}\,\left(F^{n}_{\sigma,\epsilon}\right)^{-}}, \qquad \forall K\in\mathcal{M}. \tag{1.45}$$

Under the conditions (1.42) and (1.45), the solution given by the scheme thus satisfies $\rho \geq 0$ and $e \geq 0$, and so $p \geq 0$ by the equation of state. The conservation by the scheme of the integral of the total energy over the computational domain thus yield a control on the solution.

1.4.3 Passing to the limit in the scheme (1D case)

The objective of this section is to show, in the one dimensional case, that, if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a weak formulation of the continuous problem. With the notations of Section

1.3.3, the one-dimensional version of the explicit scheme (1.40) reads:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(x)\,\mathrm{d}x, \qquad e_K^0 = \frac{1}{|K|} \int_K e_0(x)\,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\rm int}, \qquad\qquad u_\sigma^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \tag{1.46a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{1.46b}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^n e_{\sigma'}^n - F_\sigma^n e_\sigma^n + p_K^n (u_{\sigma'}^n - u_\sigma^n) = S_K^n, \tag{1.46c}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}. \tag{1.46d}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \quad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^{n+1} - p_K^{n+1} = 0, \tag{1.46e}$$

where the mass flux F_σ^n is defined by (1.11)–(1.13). In the convection terms of the internal energy balance, the approximation for e_σ^n is upwind with respect to F_σ^n (*i.e.*, for $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}$, $e_\sigma^n = e_K^n$ if $F_\sigma^n \geq 0$ and $e_\sigma^n = e_L^n$ otherwise). The corrective term S_K^n reads, $\forall K = [\sigma' \to \sigma]$:

$$S_K^{n+1} = \frac{|K|}{4\,\delta t}\,\rho_K^n \left[(u_\sigma^{n+1} - u_\sigma^n)^2 + (u_{\sigma'}^{n+1} - u_{\sigma'}^n)^2\right] + \frac{|F_K^n|}{2}\,(u_\sigma^n - u_{\sigma'}^n)^2 - |F_K^n|(u_\sigma^n - u_\sigma^n)\,(u_\sigma^n - u_{\sigma'}^n), \tag{1.47}$$

where the notation $K = [\sigma' \to \sigma]$ means that the flow goes from σ' to σ (*i.e.*, if $F_K^n \geq 0$, $K = [\overrightarrow{\sigma'\sigma}]$ and, if $F_K^n \leq 0$, $K = [\overrightarrow{\sigma\sigma'}]$).

To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual cells, so the density ρ, the pressure p, and the velocity u are defined almost everywhere on $\Omega \times (0,T)$ by (1.29), and e is defined a.e. by:

$$e(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} e_K^n\ \mathcal{X}_K(x)\ \mathcal{X}_{(n,n+1]}(t).$$

For the consistency result that we are seeking (Theorem 1.4.1 below), we have to assume that a sequence of discrete solutions $\left(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)}\right)_{m\in\mathbb{N}}$ satisfies $\rho^{(m)} \geq 0$, $p^{(m)} \geq 0$, and $e^{(m)} \geq 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability conditions (1.21), (1.42) and (1.45) are satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)^4$, *i.e.*:

$$\forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \le n \le N^{(m)}, \ \forall m \in \mathbb{N},$$
$$0 < (\rho^{(m)})^n_K \le C, \quad 0 < (p^{(m)})^n_K \le C \quad \text{and} \quad 0 < (e^{(m)})^n_K \le C, \tag{1.48}$$

and:

$$|(u^{(m)})^n_\sigma| \le C, \quad \forall \sigma \in \mathcal{E}^{(m)}, \text{ for } 0 \le n \le N^{(m)}, \ \forall m \in \mathbb{N}, \tag{1.49}$$

where C stands for a positive real number. We have to also assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|e^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{1.50}$$

and:

$$\|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{1.51}$$

A weak solution to the continuous problem satisfies, for any $\varphi \in \mathrm{C}^\infty_c\big([0,T)\times\Omega\big)$:

$$-\int_{\Omega\times(0,T)} \Big[\rho\,\partial_t\varphi + \rho\,u\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{1.52a}$$

$$-\int_{\Omega\times(0,T)} \Big[\rho\,u\,\partial_t\varphi + (\rho\,u^2 + p)\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,u_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{1.52b}$$

$$-\int_{\Omega\times(0,T)} \Big[\rho\,E\,\partial_t\varphi + (\rho\,E + p)\,u\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,E_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{1.52c}$$

$$p = (\gamma-1)\rho\,e, \quad E = \frac{1}{2}u^2 + e, \quad E_0 = \frac{1}{2}u_0^2 + e_0. \tag{1.52d}$$

As in the barotropic case, these relations are not sufficient to define a weak solution to the problem, but they allow to derive the Rankine-Hugoniot conditions; therefore, if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, we can expect that *the scheme computes correct shocks*, as stated in the following theorem.

Theorem 1.4.1 (Consistency of the one-dimensional explicit scheme, Euler case)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that ρ_0, u_0 and e_0 are functions of

$\mathrm{L}^\infty(\Omega)$. *Let* $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ *be a sequence of discretizations, which we suppose regular in the sense of Definition* 1.3.3*. Let* $(\rho^{(m)},\, p^{(m)},\, e^{(m)},\, u^{(m)})_{m\in\mathbb{N}}$ *be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (1.48)–(1.51) *and converges in* $\mathrm{L}^r((0,T)\times\Omega)^4$*, for* $1 \leq r < \infty$*, to* $(\bar\rho,\, \bar p,\, \bar e,\, \bar u) \in \mathrm{L}^\infty((0,T)\times\Omega)^4$*.*

Then the limit $(\bar\rho,\, \bar p,\, \bar e,\, \bar u)$ *satisfies the system* (1.52)*.*

Main ideas of the proof – The proof that the limit $(\bar\rho,\, \bar p,\, \bar u)$ satisfies (1.52a) and (1.52b) is the same as in the barotropic case. In addition, the fact that $(\bar\rho,\, \bar p,\, \bar e)$ satisfies the equation of state is straightforward, in view of the supposed convergence. We thus only need to prove that $(\bar\rho,\, \bar p,\, \bar e,\, \bar u)$ satisfies (1.52c). In order to do so, the technique is the same as for the proving the entropy inequality in the barotropic case. For a given smooth function φ, on one hand, we multiply the discrete kinetic energy equation (1.3) by $\delta t\, \varphi_\sigma^n$, where φ_σ^n is an interpolate of φ at the face σ and at t^n, and sum over the faces and the time steps. On the other hand, we multiply the discrete internal energy equation (1.39) by $\delta t\, \varphi_K^n$, where φ_K^n is an interpolate of φ on K at t^n, and sum over the primal cells and the time steps. Finally, summing the two obtained relations, a bit of algebra allows to conclude; we refer to Chapter 3, Theorem 3.4.2 for the detailed computation. $\square$

1.5 Radial compressible flows

In this section, we focus on the study of the barotropic and full Euler equations in case of radial explosions and implosions where blast waves propagate in radial and spherical trajectories for two and three-dimensional flows, respectively. Let us consider the system of barotropic Euler equations under the non-conservative form:

$$\partial_t \rho + \frac{1}{r^\alpha}\partial_r(r^\alpha \rho u) = 0 \tag{1.53a}$$

$$\partial_t(\rho u) + \frac{1}{r^\alpha}\partial_r(r^\alpha \rho u^2) + \partial_r p = 0 \tag{1.53b}$$

$$p = \wp(\rho) = \rho^\gamma \tag{1.53c}$$

where r is the radial direction, t is time, ρ, u and p are the density, radial velocity and pressure in the flow, and $\gamma \geq 1$ is a coefficient specific to the considered fluid. The parameter α depends on the space dimension of the problem. For $\alpha = 0$, we reproduce the one-dimensional flow which was surveyed in Section 1.3 and 1.4. The cases $\alpha = 1$

and $\alpha = 2$ correspond to the two and three-dimensional problems in cylindrical and spherical symmetry coordinate, respectively. The problem is supposed to be posed over $\Omega \times (0,T)$, where $\Omega = [0,+\infty)$ and $(0,T)$ is a finite time interval. This system must be supplemented by initial conditions for ρ and u, denoted by ρ_0 and u_0, and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition where the radial velocity vanishes at any time on $\partial\Omega$.

A weak solution to the continuous problem (1.53) satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\,\partial_t\varphi + \rho\, u\,\partial_r\varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,r^\alpha\,\mathrm{d}r = 0, \tag{1.54a}$$

$$\begin{aligned} -\int_0^T \int_\Omega \Big[\rho\, u\,\partial_t\varphi + (\rho\, u^2 + p)\,\partial_r\varphi\Big]\, r^\alpha + p\,\partial_r(r^\alpha\varphi)\,\mathrm{d}r\,\mathrm{d}t \\ - \int_\Omega \rho_0(x)\,u_0(x)\,\varphi(x,0)\,r^\alpha\,\mathrm{d}r = 0, \end{aligned} \tag{1.54b}$$

$$p = \rho^\gamma. \tag{1.54c}$$

Let us denote by E_k the kinetic energy $E_k = \frac{1}{2}\,u^2$. Taking the product of (1.54b) by u yields, after formal compositions of partial derivatives and using (1.54a):

$$\partial_t(\rho E_k) + \frac{1}{r^\alpha}\partial_r\big(r^\alpha \rho\, E_k\, u\big) + u\,\partial_r p = 0. \tag{1.55}$$

We now define the elastic potential $\mathcal{P}$, function $\mathcal{H}$ and the entropy $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$ as in Section 1.3. Multiplying the mass balance (1.53a) by $\mathcal{H}'(\rho)$ yields the elastic potential equation:

$$\partial_t\big(\mathcal{H}(\rho)\big) + \frac{1}{r^\alpha}\partial_r\big(r^\alpha \mathcal{H}(\rho)\,u\big) + \frac{1}{r^\alpha} p\,\partial_r(r^\alpha u) = 0. \tag{1.56}$$

Summing (1.55) and (1.56), we obtain the entropy equation:

$$\partial_t \mathcal{S} + \frac{1}{r^\alpha}\partial_r\big(r^\alpha(\mathcal{S} + p)\,u\big) = 0. \tag{1.57}$$

In fact, to avoid invoking unrealistic regularity assumptions, such a computation should be done on regularized equations (obtained by adding diffusion perturbation terms), and, when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (1.57), so that we get in the distribution sense:

$$\partial_t \mathcal{S} + \frac{1}{r^\alpha} \partial_r \big(r^\alpha (\mathcal{S} + p)\, u \big) \leq 0. \tag{1.58}$$

The quantity $\mathcal{S}$ is an entropy of the system, and an entropy solution to (1.53) is thus required to satisfy:

$$\begin{aligned} &\forall \varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big),\ \varphi \geq 0, \\ &\qquad \int_0^T \int_\Omega \big[-\mathcal{S} \partial_t \varphi - (\mathcal{S} + p)\, u\, \partial_r \varphi \big]\, r^\alpha \, \mathrm{d}r \, \mathrm{d}t - \int_\Omega \mathcal{S}_0\, \varphi(r,0)\, r^\alpha \, \mathrm{d}r \leq 0, \end{aligned} \tag{1.59}$$

with $\mathcal{S}_0 = \frac{1}{2} \rho_0 u_0^2 + \mathcal{H}(\rho_0)$. Then, since the radial velocity is prescribed to zero at the boundary, integrating (1.58) over Ω yields:

$$\frac{d}{dt} \int_\Omega \big[\frac{1}{2}\, \rho\, u^2 + \mathcal{H}(\rho) \big]\, r^\alpha \, \mathrm{d}r \leq 0. \tag{1.60}$$

Since $\rho \geq 0$ by (1.53a) (and the associated initial and boundary conditions) and the function $s \mapsto \mathcal{H}(s)$ is bounded by below and increasing at least for s large enough, Inequality (1.60) provides an estimate on the solution.

Let us now turn to the Euler equations on cylindrical and spherical coordinate systems under the non-conservative form:

$$\partial_t \rho + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u) = 0 \tag{1.61a}$$

$$\partial_t (\rho u) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u^2) + \partial_r p = 0 \tag{1.61b}$$

$$\partial_t (\rho E) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho E u) + \frac{1}{r^\alpha} \partial_r (r^\alpha p u) = 0 \tag{1.61c}$$

$$E = \frac{1}{2} u^2 + e \tag{1.61d}$$

$$p = (\gamma - 1) \rho e \tag{1.61e}$$

where E and e stand for the total and internal energy respectively, and $\gamma > 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0,T)$, where $\Omega = [0; +\infty)$ and $(0,T)$ is a finite time interval. Substracting the relation (1.55) from the total energy balance (1.61c), we obtain the internal energy balance equation:

$$\partial_t(\rho e) + \frac{1}{r^\alpha}\partial_r(r^\alpha \rho e u) + \frac{1}{r^\alpha}\, p\, \partial_r(r^\alpha u) = 0. \tag{1.62}$$

Since,

- thanks to the mass balance equation, the first two terms in the left-hand side of (1.62) may be recast as a transport operator: $\partial_t(\rho e)+\frac{1}{r^\alpha}\partial_r(r^\alpha \rho e u) = \rho\,[\partial_t e + u\,\partial_r e]$,
- and, from the equation of state, the pressure vanishes when $e = 0$,

this equation implies, if $e \geq 0$ at $t = 0$ and with suitable boundary conditions, that e remains non-negative at all times.

A weak solution to the continuous problem (1.61) satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\, \partial_t \varphi + \rho\, u\, \partial_r \varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \tag{1.63a}$$

$$\begin{aligned} -\int_0^T \int_\Omega \Big[\rho\, u\, \partial_t \varphi + (\rho\, u^2 + p)\, \partial_r \varphi\Big]\, r^\alpha + p\, \partial_r(r^\alpha \varphi)\,\mathrm{d}r\,\mathrm{d}t \\ - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \end{aligned} \tag{1.63b}$$

$$-\int_0^T \int_\Omega \Big[\rho\, E\, \partial_t \varphi + (\rho\, E + p)\, u\, \partial_r \varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\, E_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \tag{1.63c}$$

$$p = (\gamma - 1)\rho\, e, \quad E = \frac{1}{2}u^2 + e, \quad E_0 = \frac{1}{2}u_0^2 + e_0. \tag{1.63d}$$

The purpose of this section is to build explicit scheme(s) for the numerical solutions of System (1.53) and (1.61) and prove the following results:

- Discrete kinetic energy balance with some residual (*i.e.* a discrete analogue of (1.55)) on dual cells.
- Discrete elastic potential equation with some rest terms (*i.e.* a discrete analogue of (1.56)) on primal cells for the barotropic Euler equations. These rest terms, naturally arising from computations at discrete level, are controlled by a CFL condition to obtain the discrete version of entropy condition (1.59)
- Discrete internal energy balances with some residual (*i.e.* a discrete analogue of (4.11)) on primal cells for the Euler equations. In the contrary to rest terms in the

elastic potential equation, the residual here are imposed to complement rest terms in the discrete kinetic energy balance at the limit, when mesh size and time step tend to zero, in order to recover the total energy equation.

- Finally, passing to the limit in all equations and supposing the convergence of scheme(s), those limits are weak solutions of the continuous problem(s), and thus satisfy the Rankine-Hugoniot conditions. In particular, they are entropy solutions to the barotropic Euler equations.

1.5.1 Meshes and unknowns

All of the notations in this section are inherited from Section 1.3.3. Hereafter, we introduce new notations adapting to the cylindrical and spherical coordinate systems. The volume of K denoted by $|V_K|$ reads

$$|V_K| = \frac{r_{\sigma'}^{\alpha+1} - r_{\sigma}^{\alpha+1}}{\alpha + 1}, \qquad \forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \tag{1.64}$$

while the volume of D_σ denoted by $|V_\sigma|$ can be selected based on the way we define the dual radius r_σ. In the spirit of ISIS, the mean value of volumes of two primal cells K and L gives the volume of the dual cell D_σ

$$|V_\sigma| = \frac{|V_K| + |V_L|}{2}, \qquad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}. \tag{1.65}$$

In this way, the primal radius r_K reads

$$r_K = \sqrt[\alpha+1]{\frac{r_{\sigma}^{\alpha+1} + r_{\sigma'}^{\alpha+1}}{2}}, \qquad \forall K = [\sigma\sigma'] \in \mathcal{M}. \tag{1.66}$$

Otherwise, given $r_K = (r_\sigma + r_{\sigma'})/2$, $\forall K \in \mathcal{M}$, we define the volume of D_σ as the integral on $[r_K, r_L]$

$$|V_\sigma| = \frac{r_{L}^{\alpha+1} - r_{K}^{\alpha+1}}{\alpha + 1}, \qquad \forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}. \tag{1.67}$$

The volume of $K \cap D_\sigma$ denoted by $|V_{K,\sigma}|$, in both choices of $|V_\sigma|$, is given by

$$|V_{K,\sigma}| = \frac{|V_K|}{2}, \qquad \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}. \tag{1.68}$$

Both definitions for the volumes of dual cells, in fact, gives the same numerical solution, up to a very small tolerance, when mesh size and time step tend to zero. Therefore, in this section, we work only with the mean value volume case.

1.5.2 The barotropic Euler equations

1.5.2.1 The scheme

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(x)\,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \tag{1.69a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{1.69b}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{1.69c}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}},$$

$$\frac{|V_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + r_\sigma^\alpha\,(p_L^{n+1} - p_K^{n+1}) = 0. \tag{1.69d}$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (1.69b) is obtained by the discretization of the mass balance equation (1.53a) over the primal mesh, and F_σ^n stands for the discrete mass flux across σ outward K, which, because of the impermeability condition, vanishes on $\partial\Omega$ and is given on the internal edges by:

$$\forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad F_\sigma^n = r_\sigma^\alpha \rho_\sigma^n u_\sigma^n, \tag{1.70}$$

where the upwind approximation for the density at the edge, ρ_σ^n, is defined by

$$\rho_\sigma^n = \left| \begin{array}{ll} \rho_K^n & \text{if } u_\sigma^n \geq 0, \\ \\ \rho_L^n & \text{otherwise}. \end{array} \right. \tag{1.71}$$

We now turn to the discrete momentum balance (1.69d), which is obtained by discretizing the momentum balance equation (1.53b) on the dual cells associated to the faces of the mesh. For the discretization of the time derivative, we need to provide a definition for the values $\rho_{D_\sigma}^{n+1}$ and $\rho_{D_\sigma}^n$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\rm int}, \text{ for } k = n \text{ and } k = n+1, \qquad |V_\sigma|\ \rho_{D_\sigma}^k = |V_{K,\sigma}|\ \rho_K^k + |V_{L,\sigma}|\ \rho_L^k. \tag{1.72}$$

where $|V_{K,\sigma}| = |V_K|/2$, $\forall K \in \mathcal{M}$. The discrete mass flux F_K^n in the discretization of the convection term reads

$$\forall K = [\sigma\sigma'] \in \mathcal{M}, \qquad F_K^n = \frac{1}{2}\,(F_\sigma^n + F_{\sigma'}^n), \tag{1.73}$$

Therefore, we obtain the discrete mass balance equation on dual cells:

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}, \qquad \frac{|V_\sigma|}{\delta t}\,(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + F_L^n - F_K^n = 0, \tag{1.74}$$

Let us remark that a dual edge lying on the boundary is then also a primal edge, and the flux across that face is zero. Thanks to the discrete mass flux on dual cells, the approximation of u_K^n is given by the upwinding technique:

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad u_K^n = \left| \begin{array}{ll} u_\sigma^n & \text{if } F_K^n \geq 0, \\ \\ u_{\sigma'}^n & \text{otherwise}. \end{array} \right. \tag{1.75}$$

We denote $(\partial_r p)_\sigma^{n+1}$ and $(\partial_r u)_K^{n+1}$, respectively, the discrete derivatives of pressure at the edge σ and the velocity on primal cell K. The last term in Equation (1.69d) known as the discrete version of pressure derivative on the dual cell D_σ is built as the transpose of velocity derivative on the primal cell K. The natural approximation for the derivative of the velocity on primal cells reads

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad (\partial_r u)_K^{n+1} = \frac{1}{h_K}\big(r_{\sigma'}^{\alpha}\, u_{\sigma'}^{n+1} - r_{\sigma}^{\alpha}\, u_{\sigma}^{n+1}\big). \tag{1.76}$$

Consequently, the discrete derivative of pressure at the edge σ is given by

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \qquad (\partial_r p)_\sigma^{n+1} = \frac{1}{h_\sigma} r_\sigma^{\alpha}\, \big(p_L^{n+1} - p_K^{n+1}\big). \tag{1.77}$$

Hence, we obtain the duality relation between derivatives of pressure and velocity:

$$\sum_{K\in\mathcal{M}} h_K\, p_K^{n+1}\, (\partial_r u)_K^{n+1} + \sum_{\sigma\in\mathcal{E}_{\rm int}} h_\sigma\, u_\sigma^{n+1}\, (\partial_r p)_\sigma^{n+1} = 0. \tag{1.78}$$

Note that, because of the impermeability boundary conditions, the discrete pressure derivative is not defined at the external edges.

Finally, the initial approximations for ρ and u are given by the average of the initial conditions ρ_0 and u_0 on the primal and dual cells respectively:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|V_K|} \int_K \rho_0(r)\, r^\alpha\, \mathrm{d}r, \\ &\forall \sigma \in \mathcal{E}_{\rm int}, \quad u_\sigma^0 = \frac{1}{|V_\sigma|} \int_{D_\sigma} u_0(r)\, r^\alpha\, \mathrm{d}r. \end{aligned} \tag{1.79}$$

Note that, thanks to the upwind choice in the mass balance equation (1.71) and the assumption on the positivity of ρ_0, under the following CFL condition:

$$\delta t \leq \frac{|V_K|}{r_{\sigma'}^{\alpha}\, (u_{\sigma'}^{n})^{+} + r_{\sigma}^{\alpha}\, (u_{\sigma}^{n})^{-}}, \tag{1.80}$$

the discrete density obtained in (1.69) remains positive at all times.

1.5.2.2 Discrete kinetic energy and elastic potential balances

In the similar way to Section 1.3.2, the discrete kinetic energy equation and elastic potential balance are stated in two following lemma:

Lemma 1.5.1 (Discrete kinetic energy balance)**.**
A solution to the system (1.69) *satisfies the following equality,* $\forall n \in \{0, \ldots, N-1\}$,

$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}$, $K = \overrightarrow{\sigma'|\sigma}$, *and* $L = \overrightarrow{\sigma|\sigma''}$*:*

$$\frac{1}{2}\,\frac{|V_\sigma|}{\delta t}\Big[\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1})^2 - \rho_{D_\sigma}^{n}(u_\sigma^{n})^2\Big] + \frac{1}{2}\,\big[F_L^n\,(u_L^n)^2 - F_K^n\,(u_K^n)^2\big] + |V_\sigma|\,(\partial_r p)_\sigma^{n+1}\,u_\sigma^{n+1} = -R_\sigma^{n+1}, \quad (1.81)$$

with:

$$R_\sigma^{n+1} = \frac{1}{2}\frac{|V_\sigma|}{\delta t}\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1}-u_\sigma^n)^2 + \frac{1}{2}\big[(F_L^n)^-(u_{\sigma''}^n - u_\sigma^n)^2 + (F_K^n)^-(u_{\sigma'}^n - u_\sigma^n)^2\big] - (F_L^n)^-(u_{\sigma''}^n - u_\sigma^n)\,(u_\sigma^{n+1}-u_\sigma^n) - (F_K^n)^+(u_{\sigma'}^n - u_\sigma^n)\,(u_\sigma^{n+1}-u_\sigma^n), \quad (1.82)$$

where, for $a \in \mathbb{R}$, $a^- \geq 0$ *is defined by* $a^- = -\min(a,0)$. *This remainder term is non-negative under the following CFL condition:*

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \qquad \delta t \leq \frac{|V_\sigma|\,\rho_{D_\sigma}^{n+1}}{(F_L^n)^- + (F_K^n)^+}. \quad (1.83)$$

Lemma 1.5.2 (Discrete potential balance)**.** *Let* $\mathcal{H}$ *be defined by* (1.4)*. A solution to the system* (1.69) *satisfies the following equality, for* $K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}$, $\sigma = \overrightarrow{P|K}$, $\sigma' = \overrightarrow{K|Q}$ *and* $0 \leq n \leq N-1$*:*

$$\frac{|V_K|}{\delta t}\,\Big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\Big] + r_{\sigma'}^\alpha\,\mathcal{H}(\rho_{\sigma'}^n)\,u_{\sigma'}^n - r_\sigma^\alpha\,\mathcal{H}(\rho_\sigma^n)\,u_\sigma^n + |V_K|\,p_K^n(\partial_r u^n)_K = -R_K^{n+1}. \quad (1.84)$$

In this relation, the remainder term is defined by:

$$R_K^{n+1} = \frac{1}{2}\frac{|V_K|}{\delta t}\,\mathcal{H}''(\overline{\rho}_{K,1}^n)\,(\rho_K^{n+1}-\rho_K^n)^2 + \big(r_{\sigma'}^\alpha\,\rho_{\sigma'}^n\,u_{\sigma'}^n - r_\sigma^\alpha\,\rho_\sigma^n\,u_\sigma^n\big)\,(\rho_K^{n+1}-\rho_K^n)\,\mathcal{H}''(\overline{\rho}_{K,2}^n) + \frac{1}{2}\Big[r_{\sigma'}^\alpha\,(u_{\sigma'}^n)^-\,(\rho_Q^n - \rho_K^n)^2\,\mathcal{H}''(\overline{\rho}_{\sigma'}^n) + r_\sigma^\alpha\,(u_\sigma^n)^+\,(\rho_P^n-\rho_K^n)^2\,\mathcal{H}''(\overline{\rho}_\sigma^n)\Big], \quad (1.85)$$

with $\overline{\rho}_{K,1}^n,\ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_P^n]\!]$ *and* $\overline{\rho}_{\sigma'}^n \in [\![\rho_K^n, \rho_Q^n]\!]$, *where, for* $a,\ b \in \mathbb{R}$, *we denote by* $[\![a,b]\!]$ *the interval* $[\![a,b]\!] = \{\theta a + (1-\theta)b,\ \theta \in [0,1]\}$.

Unfortunately, it does not seem that $R_K^{n+1} \geq 0$ in any case, and so we are not able to prove a discrete counterpart of the total entropy estimate (1.60), which would yield a stability estimate for the scheme. However, under a condition for a time step which is only slightly more restrictive than a CFL-condition, and under some stability assumptions for the solutions to the scheme, we are able to show that the possible non-positive part of this remainder term tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$, which allows to conclude, in the 1D case, that a convergent sequence of solutions satisfies the entropy inequality (1.59): this is the result stated in Lemma 1.5.4 below.

1.5.2.3 Passing to the limit in the scheme

Theorem 1.5.3 (Consistency of the explicit scheme)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)},\, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (1.30)–(1.32) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^3$, for $1 \le p < \infty$, to $(\bar\rho,\, \bar p,\, \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^3$.*

Then the limit $(\bar\rho,\, \bar p,\, \bar u)$ satisfies the system (1.54)*.*

Main ideas of the proof – We refer to Chapter 4, Theorem 4.3.5 for the detail of this proof. □

Note that the discrete $\mathrm{L}^1(\Omega; \mathrm{BV}((0,T)))$ norm in this case reads:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K\in\mathcal{M}} |V_K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma\in\mathcal{E}} |V_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

We now turn to the entropy condition (1.59). To this purpose, we need to introduce the following additional condition for a sequence of discretizations:

$$\lim_{m\to+\infty} \frac{\delta t^{(m)}}{\min_{K\in\mathcal{M}^{(m)}} h_K} = 0. \tag{1.86}$$

Note that this condition is slightly more restrictive that a standard CFL condition. It allows to bound the remainder term in the discrete elastic potential balance as stated in the following lemma.

Lemma 1.5.4. *Let Ω be an open bounded interval of $\mathbb{R}$. Let $(\mathcal{M}^{(m)},\, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that the time step $\delta t^{(m)}$ tends to zero as $m \to \infty$, and*

$(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ *be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (1.30)–(1.31)*. In addition, we assume that* $(\rho^{(m)})_{m\in\mathbb{N}}$ *satisfies the following uniform BV estimate:*

$$\|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}, \tag{1.87}$$

and, for $\gamma < 2$ *only, is uniformly bounded by below,* i.e. *that there exists* $c > 0$ *such that:*

$$c \leq (\rho^{(m)})_K^n, \quad \forall K \in \mathcal{M}^{(m)},\ \textit{for}\ 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{1.88}$$

Let us suppose that the CFL condition (1.86) *hold. Let* $\mathcal{R}^{(m)}$ *be defined by:*

$$\mathcal{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} (R_K^{n+1})^-,$$

with R_K^{n+1} *given by* (1.85)*. Then:*

$$\lim_{m\to+\infty} \mathcal{R}^{(m)} = 0.$$

Then we are now in position to state the following consistency result.

Theorem 1.5.5 (Entropy consistency, barotropic case)**.** *Let the assumptions of Theorem* 1.5.3 *hold. Let us suppose in addition that the considered sequence of discretization satisfies* (1.86)*, and that* $(\rho^{(m)})_{m\in\mathbb{N}}$ *satisfies the BV estimate* (1.87) *and, for* $\gamma < 2$*, the uniform control* (1.88) *of* $1/\rho^{(m)}$ *. Then the limit* $(\bar\rho,\, \bar p,\, \bar u)$ *satisfies the entropy condition* (1.59)*.*

Main ideas of the proof – We refer to Chapter 2, Theorem 4.3.7 for the detail of this proof. □

1.5.3 The full Euler equations

1.5.3.1 The scheme

The derivation of the explicit-in-time scheme for the Euler equations is obtained in the same manner to the barotropic Euler equations (Section 1.5.2.1). The fully discrete form of the scheme reads, for $0 \leq n \leq N-1$:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|V_K|} \int_K \rho_0(x)\, r^\alpha \,\mathrm{d}r, \qquad e_K^0 = \frac{1}{|V_K|} \int_K e_0(x)\, r^\alpha \,\mathrm{d}r, \\ &\forall \sigma \in \mathcal{E}_{\rm int}, \quad u_\sigma^0 = \frac{1}{|V_\sigma|} \int_{D_\sigma} u_0(x)\, r^\alpha \,\mathrm{d}r, \end{aligned} \tag{1.89a}$$

$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$,

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{1.89b}$$

$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$,

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^n e_{\sigma'}^n - F_\sigma^n e_\sigma^n + p_K^n (r_{\sigma'}^\alpha u_{\sigma'}^n - r_\sigma^\alpha u_\sigma^n) = S_K^n, \tag{1.89c}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}, \tag{1.89d}$$

$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}$,

$$\frac{|V_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + r_\sigma^\alpha\, (p_L^{n+1} - p_K^{n+1}) = 0. \tag{1.89e}$$

Equations (1.89b) and (1.89e) are introduced in Section 1.5.2.1. Equation (1.89c) is an approximation of the internal energy balance over the primal cell K. The positivity of the convection operator is ensured thanks to the upwinding choice for e_σ^n:

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \qquad e_\sigma^n = \left| \begin{array}{ll} e_K^n & \text{if } F_\sigma^n \geq 0, \\ e_L^n & \text{otherwise.} \end{array} \right.$$

The last term on the left-hand side is a natural approximation of the velocity derivative on primal cells which is given by (4.27). The right-hand side, S_K^n, is derived by using consistency arguments in the next section. Finally, the initial approximations for e is given by the average of the initial conditions e_0 on the primal cells.

1.5.3.2 Corrective source terms

With the same idea as in Section 1.4.2, the source term (S_K^n) is chosen to compensate the terms (R_σ^{n+1}) given by (1.85):

$$\forall K \in \mathcal{M},\ K = [\overrightarrow{\sigma\sigma'}], \qquad S_K^n = \frac{|V_K|}{4\,\delta t}\,\rho_K^n\,\big[(u_\sigma^n - u_\sigma^{n-1})^2 + (u_{\sigma'}^n - u_{\sigma'}^{n-1})^2\big]$$
$$+ \frac{|F_K^{n-1}|}{2}\,(u_\sigma^{n-1} - u_{\sigma'}^{n-1})^2 + F_K^{n-1}\,(u_{\sigma'}^{n-1} - u_\sigma^{n-1})\,(u_K^n - u_K^{n-1}), \quad (1.90)$$

where $u_K^n - u_K^{n-1}$ is a downwind choice with respect to F_K^{n-1}:

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad u_K^n - u_K^{n-1} = \left| \begin{array}{ll} u_{\sigma'}^n - u_{\sigma'}^{n-1} & \text{if } F_K^{n-1} \geq 0, \\ u_\sigma^n - u_\sigma^{n-1} & \text{otherwise}. \end{array} \right.$$

The definition of $(S_K^n)_{K\in\mathcal{M}}$ allows to prove that, under a CFL condition, the scheme preserves the positivity of e.

Lemma 1.5.6. *Let us suppose that, for $1 \leq n \leq N$ and for all $K = \sigma|\sigma' \in \mathcal{M}$, we have:*

$$\delta t \leq \frac{|V_K|}{\gamma\big[r_{\sigma'}^\alpha\ (u_{\sigma'}^n)^+ + r_\sigma^\alpha\ (u_\sigma^n)^-\big]} \quad \textit{and} \quad \delta t \leq \frac{|V_K|\ \rho_K^n}{\big|F_\sigma^{n-1} + F_{\sigma'}^{n-1}\big|}. \qquad (1.91)$$

Then the internal energy $(e^n)_{0\leq n\leq N}$ given by the scheme (1.89) *is positive.*

1.5.3.3 Passing to the limit in the scheme

Theorem 1.5.7 (Consistency of the explicit scheme)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$, $p_0 \in \mathrm{BV}(\Omega)$, $e_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and let $(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (4.45)–(4.48) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^4$, for $1 \leq p < \infty$, to $(\bar\rho,\ \bar p,\ \bar e,\ \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^4$.*

Then the limit $(\bar\rho,\ \bar p,\ \bar e,\ \bar u)$ satisfies the system (1.63)*.*

Main ideas of the proof – We refer to Chapter 3, Theorem 4.4.2 for the detail of this proof. □

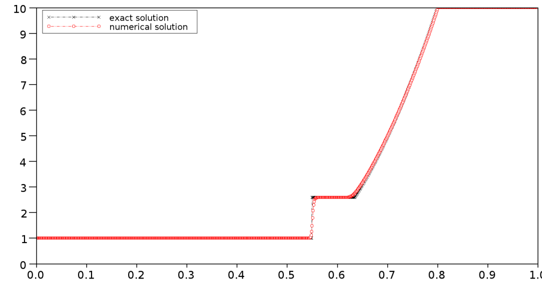
1.6 Some numerical results

We test here the explicit schemes on some one dimensional model problems.

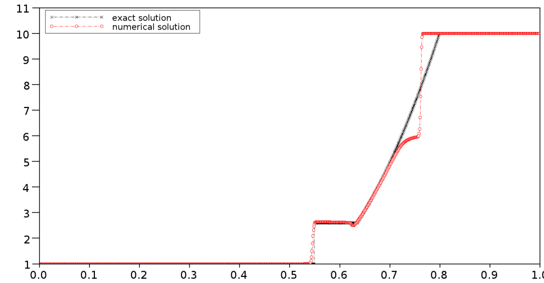
1.6.1 The barotropic case

In this section, we give some numerical results for the barotropic equations, taking $p = \rho^2$ as equation of state. Note that in this case, the system is equivalent (up to a constant proportionality coefficient in the equation of state) to the shallow water equations, replacing ρ by the water height h. We test the explicit scheme studied above, which we denote by $\rho \to p \to u$, since the pressure is updated before solving the momentum balance equation; we also test the "naive" explicit scheme obtained by evaluating all terms other than the time derivative at time t^n, which we denote by $\rho \to u \to p$ (the pressure is then updated after the computation of the velocity rather than after the computation of the density).

We consider a one–dimensional Riemann problem with the following values for the left and right states: $\rho_L = 1.$, $\rho_R = 10.$, $u_L = 5.$ and $u_R = 7.5$, which yields a solution with a left shock and a right rarefaction wave. We plot the computed density and velocity at time $T = 0.025$ on figures 1.2 and 1.3 respectively. From these results, it appears clearly that the so-called "naive" scheme generates discontinuities in the rarefaction wave, and further experiments show that this phenomenon is not cured by a reduction of the time and space step; this seems to be connected to the fact that, for this variant, we cannot prove that the limits of converging sequences satisfy the entropy condition (and they probably do not). When trying to do so, in our proof and from a purely technical point of view, the trouble comes from the fact that the pressure gradient term which appears in the kinetic energy balance reads $\boldsymbol{u}^{n+1}\boldsymbol{\nabla} p^n$ and it seems difficult to make the counterpart (*i.e.* $p^n \mathrm{div}(\boldsymbol{u}^{n+1})$), with the corresponding time levels, appear in the elastic potential balance, starting from a mass balance with a convection term written with $\boldsymbol{u}^n$; hence a discretization of the momentum balance equation with an updated pressure gradient term $\boldsymbol{\nabla} p^{n+1}$, and thus the inversion of steps in the algorithm, to get the "reference variant" proposed in this chapter. This latter scheme seems to converge to the right solution, and it is confirmed by a numerical convergence study letting the space and time steps tend to zero, which show an approximatively first-order rate of convergence.

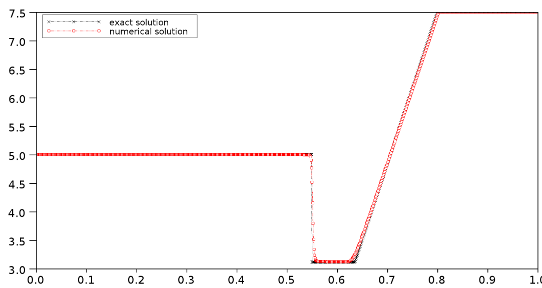


(a) $\rho \to p \to u$

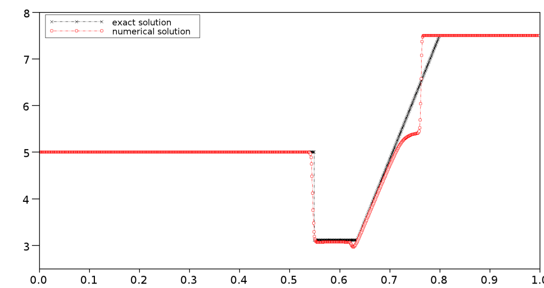


(b) $\rho \to u \to p$

Figure 1.2: Barotropic Euler equations, density



(a) $\rho \to p \to u$



(b) $\rho \to u \to p$

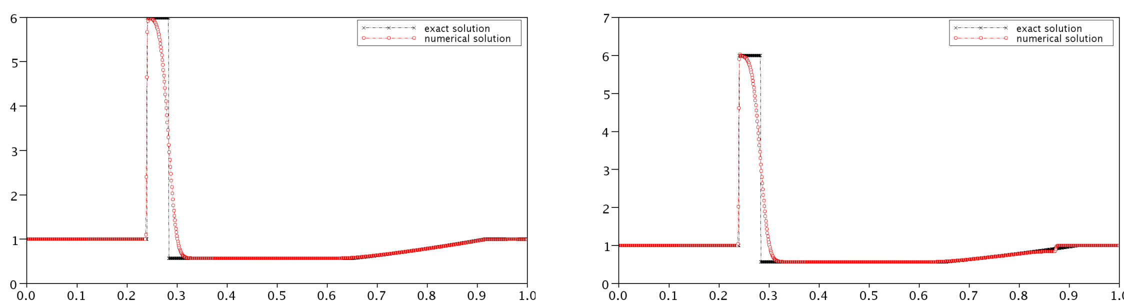
Figure 1.3: Barotropic Euler equations, velocity

1.6.2 The full Euler equations

Let us now turn to the full Euler equations. We use the test case referred as Test 4 in [61, Chapter 4], which is a Riemann problem with the following initial states: $\rho_L = \rho_R = 1.$, $u_L = u_R = 0.$, $p_L = 0.01$ and $p_R = 100$. Here again, we test the scheme that was analysed in Section 1.4, which we denote by $\rho \to e \to p \to u$ (the order in which we compute the unknowns at time $n+1$) and compare it to the “naive” scheme, denoted by $\rho \to u \to u \to p$, obtained by discretizing the Euler equations (1.38) in the corresponding order.

The density, velocity, internal energy and pressure obtained at $T = 0.035$, together with the analytical solution, are plotted on Figures 1.4, 1.5, 1.6 and 1.7 respectively. For the naive scheme, the same behaviour as for the barotropic case (*i.e.* the presence of discontinuities in the rarefaction wave) is observed, while, once again, the reference variant of the algorithm yields correct results. The order of the scheme is numerically found to be 1 for the variables with no jump at the discontinuities (these are p and u) and 1/2 for those with a jump, namely ρ and e. However, the diffusive character of the scheme is evidenced at the contact discontinuity; the implementation of a more accurate discretization, based on a MUSCL-like technique, is underway.

In addition, we also tested the scheme obtained by neglecting the corrective terms $(S_K)_{K\in\mathcal{M}}$ in the internal energy balance; results (not plotted here) seem to show that this scheme does converge, but toward a limit which is clearly not a weak solution to the Euler equations (in particular, with jumps which do not satisfy the Rankine-Hugoniot conditions).

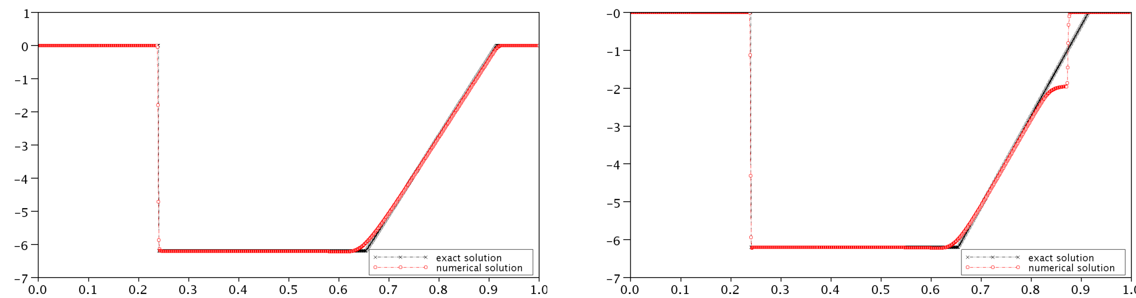


(a) $\rho \to e \to p \to u$ (b) $\rho \to u \to e \to p$

Figure 1.4: Full Euler equations, density

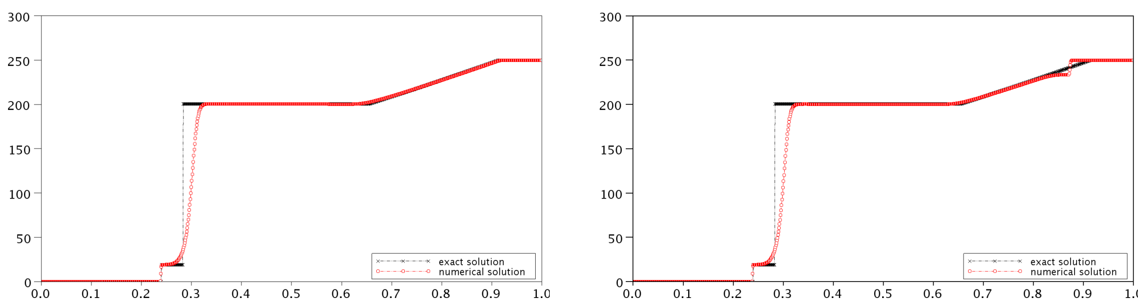
1.6.3 Radial compressible flows

We address the Riemann problem studied in [61, Chapter 17] to assess the behaviour of the scheme on the explosion. The chosen initial states $\rho_{ins} = 1$, $u_{ins} = 0$; $p_{ins} = 1$, $\rho_{out} = 0.125$, $u_{out} = 0$ and $p_{out} = 0.1$ gives a circular shock wave travelling away from the centre, a circular contact surface travelling in the same direction and a circular rarefaction travelling towards the origin. The three-dimensional solutions Figure 1.8 including density, velocity, pressure and internal energy obtained along the radial line that is coincident with the x–axis at the final time are compatible with the reference solutions in [61, Figure 17.7].



(a) $\rho \to e \to p \to u$ (b) $\rho \to u \to e \to p$

Figure 1.5: Full Euler equations, velocity



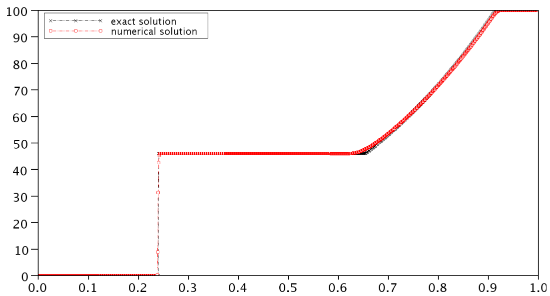
(a) $\rho \to e \to p \to u$ (b) $\rho \to u \to e \to p$

Figure 1.6: Full Euler equations, internal energy

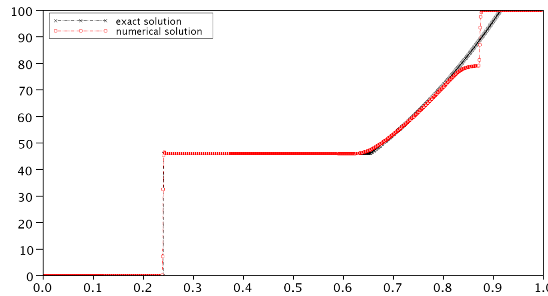
1.7 Conclusion

We present in this thesis an explicit scheme based on staggered meshes for the hyperbolic system of the compressible flows. This algorithm uses a very simple first-order upwinding strategy which consists, equation by equation, to implement an upwind discretization with respect of the material velocity of the convection term.

- For the barotropic Euler equations: under CFL-like conditions based on the material velocity only (by opposition to the celerity of waves), our scheme preserves the positivity of the density and the pressure, and has been shown to be consistent for 1D problems, in the sense that, if a sequence of numerical solutions obtained with more and more refined meshes (and, accordingly, smaller and smaller time steps) converges, then the limit is a weak entropy solution to the continuous problem.



(a) $\rho \to e \to p \to u$

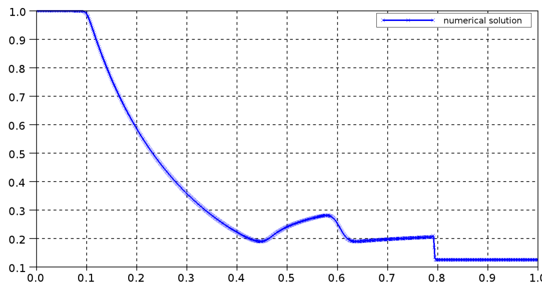


(b) $\rho \to u \to e \to p$

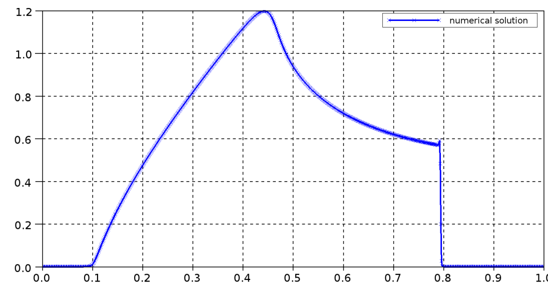
Figure 1.7: Full Euler equations, pressure

- For the full Euler equations: our scheme solves the internal energy balance instead of the total energy balance, and thus turns out to be non-conservative: indeed, the total energy conservation law is only recovered at the limit of vanishing time and space steps, thanks to the addition of corrective source terms in the discrete internal energy balance. Under CFL-like conditions based on the material velocity only, this scheme preserves the positivity of the density, the internal energy and the pressure (in other words, the scheme preserves the convex of admissible states), and its solution satisfies a conservation property (in fact, as often at the discrete level, non-increase) of the integral of the total energy over the computational domain. Finally, the scheme has been shown to be consistent for 1D problems, in the sense that, if a sequence of numerical solutions obtained with more and more refined meshes (and, accordingly, smaller and smaller time steps) converges, then the limit is a weak solution to the continuous problem.

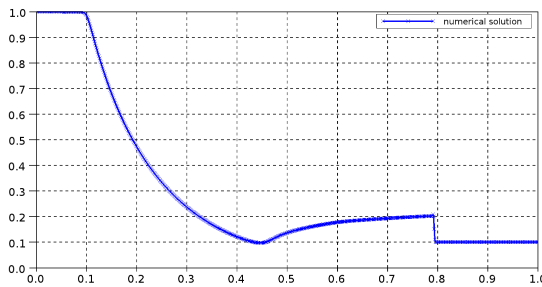
- For the radial compressible flows: this case is, in fact, an extension of 1D flows on two and three-dimensional spaces where the acoustic waves propagate in radial and spherical trajectories. From the theoretical and numerical point of views, our scheme still gives the same properties as stated above for the barotropic and full Euler equations. The obtained results depend only on the volume and the connectivity of the mesh. Therefore, our scheme can apply on wider classes of problems, for instance, the axisymmetric flow with non-zero angular velocity component.



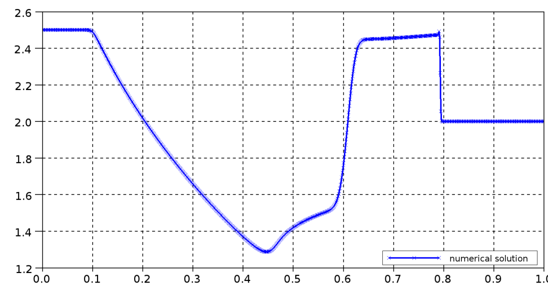
(a) density



(b) velocity



(c) pressure



(d) internal energy

Figure 1.8: Euler equations on spherical coordinate systems, all solutions

Numerical studies show that the proposed algorithm is stable, even if the largest time step before blow-up is smaller than suggested by the above-mentioned CFL conditions. This behaviour had to be expected, since these CFL conditions only involve the velocity (and not the celerity of the acoustic waves): indeed, was this the only limitation, we would have obtained an explicit scheme stable up to the incompressible limit. However, the mechanisms leading to the blow-up of the scheme (or, conversely, the way to fix the time step to ensure stability) remain to be understood.

In addition, numerical experiments show that some oscillations appear near stagnation points, where the numerical diffusion brought by the upwinding vanishes. These oscillations are damped by a small amount of artificial (physical-like) viscosity, and this suggests to implement techniques consisting in adding to the scheme such a diffusion term, with a viscosity monitored by an *a posteriori* (*i.e.* performed in view of the results of the previous time step) analysis of the solution, as the so-called entropy-viscosity technique. Besides, such an extension should allow to design a more accurate scheme, based on higher-order numerical fluxes. This work is underway.

Since the proposed scheme uses very simple numerical fluxes, it is well suited to large multi-dimensional parallel computing applications, and such studies are now beginning at IRSN. Still for the same reasons (and, in particular, because the construction of the discretization does not require the solution of the Riemann problem), it seems that the presented approach offers natural extensions to more complex problems, such as reacting flows; this development is foreseen at IRSN, for applications to explosion hazards.

Chapter 2

The barotropic Euler equations

2.1 Introduction

We address in the work the numerical solution of the so-called barotropic Euler equations, which read:

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{2.1a}$$

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \tag{2.1b}$$

$$p = \wp(\rho) = \rho^\gamma, \tag{2.1c}$$

where t stands for the time, ρ, $\boldsymbol{u}$ and p are the density, velocity and pressure in the flow, and $\gamma \geq 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0,T)$, where Ω is an open bounded connected subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, and $(0,T)$ is a finite time interval. This system must be supplemented by initial conditions for ρ and $\boldsymbol{u}$, denoted by ρ_0 and $\boldsymbol{u}_0$, and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition, which we suppose to be:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$

at any time and *a.e.* on $\partial\Omega$, where $\boldsymbol{n}$ stands for the normal vector to the boundary.

Let us denote by E_k the kinetic energy $E_k = \frac{1}{2} |\boldsymbol{u}|^2$. Taking the inner product of (2.1b) by $\boldsymbol{u}$ yields, after formal compositions of partial derivatives and using the mass balance (2.1a):

$$\partial_t(\rho E_k) + \mathrm{div}\big(\rho\, E_k\, \boldsymbol{u}\big) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0. \tag{2.2}$$

This relation is referred to as the kinetic energy balance.

Let us now define the function $\mathcal{P}$, from $(0, +\infty)$ to $\mathbb{R}$, as a primitive of $s \mapsto \wp(s)/s^2$; this quantity is often called the elastic potential. Let $\mathcal{H}$ be the function defined by $\mathcal{H}(s) = s\mathcal{P}(s),\ \forall s \in (0, +\infty)$. For the specific equation of state $\wp$ used here, we obtain:

$$\mathcal{H}(s) = s\,\mathcal{P}(s) = \begin{cases} \dfrac{s^\gamma}{\gamma - 1} & \text{if } \gamma > 1, \\[2ex] s \ln(s) & \text{if } \gamma = 1. \end{cases} \tag{2.3}$$

Since $\wp$ is an increasing function, $\mathcal{H}$ is convex. In addition, it may easily be checked that $\rho\mathcal{H}'(\rho) - \mathcal{H}(\rho) = \wp(\rho)$. Therefore, by a formal computation, detailed in the appendix, multiplying (2.1a) by $\mathcal{H}'(\rho)$ yields:

$$\partial_t\big(\mathcal{H}(\rho)\big) + \mathrm{div}\big(\mathcal{H}(\rho)\, \boldsymbol{u}\big) + p\, \mathrm{div}(\boldsymbol{u}) = 0. \tag{2.4}$$

Let us denote by $\mathcal{S}$ the quantity $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$. Summing (2.2) and (2.4), we get:

$$\partial_t \mathcal{S} + \mathrm{div}\big((\mathcal{S} + p)\, \boldsymbol{u}\big) = 0. \tag{2.5}$$

In fact, to avoid invoking unrealistic regularity assumption, such a computation should be done on regularized equations (obtained by adding diffusion perturbation terms), and, when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (2.5), so that we get in the distribution sense:

$$\partial_t \mathcal{S} + \mathrm{div}\big((\mathcal{S} + p)\, \boldsymbol{u}\big) \le 0. \tag{2.6}$$

The quantity $\mathcal{S}$ is an entropy of the system, and an entropy solution to (2.1) is thus required to satisfy:

$$\begin{aligned} &\forall \varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0, T)\big),\ \varphi \ge 0, \\ &\qquad \int_0^T \int_\Omega \big[-\mathcal{S}\partial_t\varphi - (\mathcal{S} + p)\, \boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi\big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t - \int_\Omega \mathcal{S}_0\, \varphi(\boldsymbol{x}, 0)\, \mathrm{d}\boldsymbol{x} \le 0, \end{aligned} \tag{2.7}$$

with $\mathcal{S}_0 = \frac{1}{2}\rho_0|\boldsymbol{u}_0|^2 + \mathcal{H}(\rho_0)$. Then, since the normal velocity is prescribed to zero at the boundary, integrating (2.6) over Ω yields:

$$\frac{d}{dt}\int_\Omega \big[\frac{1}{2}\,\rho\,|\boldsymbol{u}|^2 + \mathcal{H}(\rho)\big]\,\mathrm{d}\boldsymbol{x} \leq 0. \tag{2.8}$$

Since $\rho \geq 0$ by (2.1a) (and the associated initial and boundary conditions) and the function $s \mapsto \mathcal{H}(s)$ is bounded by below and increasing at least for s large enough, Inequality (2.8) provides an estimate on the solution.

The purpose of this chapter is to build an explicit scheme for the numerical solution of System (2.1). This scheme is, in fact, the explicit variant of a recent all-Mach-number pressure correction scheme [15, 26] implemented in the open-source software ISIS [33], and is developed with the aim to offer an efficient alternative for quickly varying unstationary flows, with a characteristic Mach number in the range or greater than the unity. The proposed algorithm thus keeps the space discretizations used in this context, namely staggered finite volume or finite element discretizations. This discretization precludes the use of Riemann solvers (see *e.g.* [61, 19, 6] for textbooks on this latter technique), and we thus implement the most naive upwinding, with respect to the material velocity only (similarly, but with a simpler upwinding algorithm, to what is proposed in the collocated context in the AUSM method [45, 44]). The pressure gradient is defined as the transpose of the natural velocity divergence, and is thus centered. Last but not least, the velocity convection term is built in such a way to allow to derive a discrete kinetic energy balance.

We prove for this scheme the following results:

- a discrete kinetic energy balance (*i.e.* a discrete analogue of (2.2)) is established on dual cells, while a discrete potential elastic balance (*i.e.* a discrete analogue of (2.4)) is established on primal cells.
  Note however that, because of residual terms appearing in the potential elastic balance, contrary to what is obtained for implicit and semi-implicit variants of the present scheme [15, 26], these equations do not seem to yield the stability of the scheme (*i.e.* a discrete global entropy conservation analogue to Equation (2.8)), at least unless supposing drastic limitations of the time step.

- Second, in one space dimension, the limit of any convergent sequence of solutions to the scheme is shown to be a weak solution to the continuous problem, and thus to satisfy the Rankine-Hugoniot conditions.

- Finally, passing to the limit in the discrete kinetic energy and elastic potential balances, such a limit is also shown to satisfy the entropy inequality (2.7).

This chapter is structured as follows. We begin with the presentation the scheme (Section 2.2), then the discrete kinetic and elastic potential balances are given in Section 2.3. The next section is dedicated to the proof, in 1D, of the consistency of the scheme (Section 2.4). We then present some numerical tests, to assess the behaviour of the algorithm (Section 2.5). The discrete kinetic energy and elastic potential balances are obtained as particular cases of more general results applying to the explicit finite volume discretization of transport operators, which are established in Appendix 2.7. Finally, the conclusion and perspectives are given in Section 2.6.

2.2 The scheme

We refer to Chapter 1, Section 1.2 for the space discretization. For the discretization in time, let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n = 0, \tag{2.9a}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma, \tag{2.9b}$$

For $1 \leq i \leq d,\ \forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$,

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n u_{\epsilon,i}^n + |D_\sigma|\,(\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0, \tag{2.9c}$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (2.9a) is obtained by the discretization of the mass balance equation (2.1a) over the primal mesh, and $F_{K,\sigma}^{n+1}$ stands for the mass flux across σ outward K, which, because of the impermeability condition, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad F^n_{K,\sigma} = |\sigma|\ \rho^n_\sigma\ u^n_{K,\sigma}, \tag{2.10}$$

where $u^n_{K,\sigma}$ is an approximation of the normal velocity to the face σ outward K. This latter quantity is defined by:

$$u^n_{K,\sigma} = \left| \begin{array}{l} u^n_{\sigma,i}\ \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\[2ex] \boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{K,\sigma} \text{ in the CR and RT cases,} \end{array} \right. \tag{2.11}$$

where $\boldsymbol{e}^{(i)}$ denotes the i-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face $\sigma = K|L$ is approximated by the upwind technique:

$$\rho^n_\sigma = \left| \begin{array}{ll} \rho^n_K & \quad \text{if } u^n_{K,\sigma} \geq 0, \\[2ex] \rho^n_L & \quad \text{otherwise}. \end{array} \right. \tag{2.12}$$

We now turn to the discrete momentum balance (2.9c), which is obtained by discretizing the momentum balance equation (2.1b) on the dual cells associated to the faces of the mesh. For the discretization of the time derivative, we must provide a definition for the values $\rho^{n+1}_{D_\sigma}$ and $\rho^n_{D_\sigma}$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\rm int},\ \text{for } k = n \text{ and } k = n+1, \qquad |D_\sigma|\ \rho^k_{D_\sigma} = |D_{K,\sigma}|\ \rho^k_K + |D_{L,\sigma}|\ \rho^k_L. \tag{2.13}$$

Let us then turn to the discretization of the convection term. The first task is to define the discrete mass flux through the dual face ϵ outward D_σ, denoted by $F^n_{\sigma,\epsilon}$; the guideline for its construction is that a finite volume discretization of the mass balance equation over the diamond cells, of the form

$$\forall \sigma \in \mathcal{E}, \qquad \frac{|D_\sigma|}{\delta t}\ (\rho^{n+1}_{D_\sigma} - \rho^n_{D_\sigma}) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F^n_{\sigma,\epsilon} = 0, \tag{2.14}$$

must hold in order to be able to derive a discrete kinetic energy balance (see Section 2.3 below). For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [30]. In the case of the CR and RT schemes, for a dual face ϵ included in the primal cell K, this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the face and the cell) of the mass fluxes through the faces of K, *i.e.* the quantities $(F_{K,\sigma}^{n+1})_{\sigma\in\mathcal{E}(K)}$ appearing in the discrete mass balance (2.9a). We refer to [1, 17] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across that face is zero. Therefore, the values $u_{\epsilon,i}^{n+1}$ are only needed at the internal dual faces, and we make the upwind choice for their discretization:

$$\text{for } \epsilon = D_\sigma|D'_\sigma, \quad u_{\epsilon,i}^n = \left| \begin{array}{ll} u_{\sigma,i}^n & \text{if } F_{\sigma,\epsilon}^n \geq 0, \\ \\ u_{\sigma',i}^n & \text{otherwise.} \end{array} \right. \tag{2.15}$$

The last term $(\boldsymbol{\nabla} p)_{\sigma,i}^{n+1}$ stands for the i-th component of the discrete pressure gradient at the face σ. The gradient operator is built as the transpose of the discrete operator for the divergence of the velocity, the discretization of which is based on the primal mesh. Let us denote the divergence of $\boldsymbol{u}^{n+1}$ over $K \in \mathcal{M}$ by $(\mathrm{div}\boldsymbol{u})_K^{n+1}$; its natural approximation reads:

$$\text{for } K \in \mathcal{M}, \qquad (\mathrm{div}\boldsymbol{u})_K^{n+1} = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, u_{K,\sigma}^{n+1}. \tag{2.16}$$

Consequently, the components of the pressure gradient are given by:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = \frac{|\sigma|}{|D_\sigma|}\, (p_L^{n+1} - p_K^{n+1})\, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}, \tag{2.17}$$

this expression being derived thanks to the following duality relation with respect to the L^2 inner product:

$$\sum_{K\in\mathcal{M}} |K|\, p_K^{n+1}\, (\mathrm{div}\boldsymbol{u})_K^{n+1} + \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}_{\mathcal{S}}^{(i)}} |D_\sigma|\, u_{\sigma,i}^{n+1}\, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0. \tag{2.18}$$

Note that, because of the impermeability boundary conditions, the discrete gradient is not defined at the external faces.

Finally, the initial approximations for ρ and $\boldsymbol{u}$ are given by the average of the initial conditions ρ_0 and $\boldsymbol{u}_0$ on the primal and dual cells respectively:

$$\begin{aligned}
&\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \\
&\text{for } 1 \leq i \leq d,\ \forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \quad u_{\sigma,i}^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} (\boldsymbol{u}_0(\boldsymbol{x}))_i \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{2.19}$$

The following positivity result is a classical consequence of the upwind choice in the mass balance equation.

Lemma 2.2.1 (Positivity of the density)**.** *Let ρ^0 be given by* (2.19)*. Then, since u_0 is assumed to be a positive function, $\rho^0 > 0$ and, under the* CFL *condition:*

$$\delta t \leq \frac{|K|}{\sum_{\sigma\in\mathcal{E}(K)} |\sigma| \ \max(u_{K,\sigma}^n, 0)}, \qquad \forall K \in \mathcal{M},\ \textit{for } 0 \leq n \leq N-1, \tag{2.20}$$

the solution to the scheme satisfies $\rho^n > 0$, for $1 \leq n \leq N$.

2.3 Discrete kinetic energy and elastic potential balances

We begin by deriving a discrete kinetic energy balance equation, as was already done for the implicit and fractional time step scheme described in [26]. Equation (2.21) is a discrete analogue of Equation (2.2), with an upwind discretization of the convection term.

Lemma 2.3.1 (Discrete kinetic energy balance)**.**
A solution to the system (2.9) *satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$ and $0 \leq n \leq N-1$:*

$$\frac{1}{2}\,\frac{|D_\sigma|}{\delta t}\Big[\rho_{D_\sigma}^{n+1}(u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n}(u_{\sigma,i}^{n})^2\Big] + \frac{1}{2}\sum_{\epsilon\in\mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n\,(u_{\epsilon,i}^n)^2 + |D_\sigma|\,(\boldsymbol{\nabla} p)_{\sigma,i}^{n+1}\,u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \tag{2.21}$$

with:

$$R_{\sigma,i}^{n+1} = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\rho_{D_\sigma}^{n+1}(u_{\sigma,i}^{n+1} - u_{\sigma,i}^{n})^2 + \frac{1}{2}\sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F_{D_\sigma,\epsilon}^n)^-(u_{\sigma',i}^n - u_{\sigma,i}^n)^2 \\ - \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F_{D_\sigma,\epsilon}^n)^-(u_{\sigma',i}^n - u_{\sigma,i}^n)\,(u_{\sigma,i}^{n+1} - u_{\sigma,i}^n), \quad (2.22)$$

where, for $a \in \mathbb{R}$, $a^- \geq 0$ is defined by $a^- = -\min(a,0)$. This remainder term is non-negative under the following CFL condition:

$$\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \qquad \delta t \leq \frac{|D_\sigma|\ \rho_{D_\sigma}^{n+1}}{\sum_{\epsilon\in\mathcal{E}(D_\sigma)}(F_{\sigma,\epsilon}^n)^-}. \quad (2.23)$$

Proof. The proof of this lemma is obtained by multiplying the (i^{th} component of the) momentum balance equation (2.9c) associated to the face σ by the unknown $u_{\sigma,i}^{n+1}$, and invoking Lemma 2.7.2 of the appendix. ☐

Similarly, the solution to the scheme (2.9) satisfies a discrete version of the elastic potential identity (2.4), which we now state.

Lemma 2.3.2 (Discrete potential balance)**.** *Let $\mathcal{H}$ be defined by* (2.3)*. A solution to the system* (2.9) *satisfies the following equality, for $K \in \mathcal{M}$ and $0 \leq n \leq N-1$:*

$$\frac{|K|}{\delta t}\left[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\right] + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\mathcal{H}(\rho_\sigma^n)\,u_{K,\sigma}^n + |K|\,p_K^n(\mathrm{div}\boldsymbol{u}^n)_K = -R_K^{n+1}. \quad (2.24)$$

In this relation, the remainder term is defined by:

$$R_K^{n+1} = \frac{1}{2}\frac{|K|}{\delta t}\,\mathcal{H}''(\overline{\rho}_{K,1}^n)\,(\rho_K^{n+1} - \rho_K^n)^2 + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,u_{K,\sigma}^n\,\mathcal{H}''(\overline{\rho}_{K,2}^n)\,\rho_\sigma^n\,(\rho_K^{n+1} - \rho_K^n) \\ + \frac{1}{2}\sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\,(u_{K,\sigma}^n)^-\,\mathcal{H}''(\overline{\rho}_\sigma^n)\,(\rho_K^n - \rho_L^n)^2, \quad (2.25)$$

with $\overline{\rho}_{K,1}^n,\ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, and $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_L^n]\!]$ for all $\sigma \in \mathcal{E}(K)$, where, for $a,\ b \in \mathbb{R}$, we denote by $[\![a,b]\!]$ the interval $[\![a,b]\!] = \{\theta a + (1-\theta)b,\ \theta \in [0,1]\}$.

Proof. The proof of this lemma is obtain by multiplying the discrete mass balance equation (2.9a) by $\mathcal{H}'(\rho_K^{n+1})$ and invoking Lemma 2.7.1 of the appendix. ☐

Unfortunately, it does not seem that $R_K^{n+1} \geq 0$ in any case, and so we are not able to prove a discrete counterpart of the total entropy estimate (2.8), which would yield a stability estimate for the scheme. However, under a condition for a time step which is only slightly more restrictive than a CFL-condition, and under some stability assumptions for the solutions to the scheme, we are able to show that the possible non-positive part of this remainder term tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$, which allows to conclude, in the 1D case, that a convergent sequence of solutions satisfies the entropy inequality (2.7): this is the result stated in Lemma 2.4.3 below.

2.4 Passing to the limit in the scheme

The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

The 1D version of the scheme which is studied in this section may be obtained from Scheme (2.9) by taking the MAC variant of the scheme, using only one horizontal stripe of grid cells, supposing that the vertical component of the velocity (the degrees of freedom of which are located on the top and bottom boundaries) vanishes, and that the measure of the vertical faces is equal to 1. For the sake of readability, however, we completely rewrite this 1D scheme, and, to this purpose, we first introduce some adaptations of the notations to the one dimensional case. For any $K \in \mathcal{M}$, we denote by h_K its length (so $h_K = |K|$); when we write $K = [\sigma\sigma']$, this means that either $K = (x_\sigma, x_{\sigma'})$ or $K = (x_{\sigma'}, x_\sigma)$; if we need to specify the order, *i.e.* $K = (x_\sigma, x_{\sigma'})$ with $x_\sigma < x_{\sigma'}$, then we write $K = [\overrightarrow{\sigma\sigma'}]$. For an interface $\sigma = K|L$ between two cells K and L, we define $h_\sigma = (h_K + h_L)/2$, so, by definition of the dual mesh, $h_\sigma = |D_\sigma|$. If we need to specify the order of the cells K and L, say K is left of L, then we write $\sigma = \overrightarrow{K|L}$. With these notations, the explicit scheme (2.9) may be written as follows in the one dimensional setting:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(x)\,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \tag{2.26a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$
$$\frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{2.26b}$$
$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{2.26c}$$
$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int},$$
$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^{n+1} - p_K^{n+1} = 0, \tag{2.26d}$$

The mass flux in the discrete mass balance equation is given, for $\sigma \in \mathcal{E}_{\rm int}$, by $F_\sigma^n = \rho_\sigma^n u_\sigma^n$, where the upwind approximation for the density at the face, ρ_σ^n, is defined by (2.12). In the momentum balance equation, the application of the procedure described in Section 2.2 yields for the density associated to the dual cell D_σ with $\sigma = K|L$ and for the mass fluxes at the dual face located at the center of the mesh $K = [\overrightarrow{\sigma\sigma'}]$:

$$\text{for } k = n \text{ and } k = n+1,\ \rho_{D_\sigma}^k = \frac{1}{2\,|D_\sigma|}\,(|K|\,\rho_K^k + |L|\,\rho_L^k), \qquad F_K^n = \frac{1}{2}\,(F_\sigma^n + F_{\sigma'}^n), \tag{2.27}$$

and the approximation of the velocity at this face is upwind: $u_K^n = u_\sigma^n$ if $F_K^n \geq 0$ and $u_K^n = u_{\sigma'}^n$ otherwise.

Let a sequence of discretizations $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ be given. We define the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ by $h^{(m)} = \sup_{K\in\mathcal{M}^{(m)}} h_K$. Let $\rho^{(m)}$, $p^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (2.26) with the mesh $\mathcal{M}^{(m)}$ and the time step $\delta t^{(m)}$. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho^{(m)}$, the pressure $p^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0,T)$ by:

$$\rho^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (\rho^{(m)})_K^n\, \mathcal{X}_K(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{2.28}$$
$$u^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} (u^{(m)})_\sigma^n\, \mathcal{X}_{D_\sigma}(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{2.29}$$
$$p^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (p^{(m)})_K^n\, \mathcal{X}_K(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{2.30}$$

where $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{(n,n+1]}$ stand for the characteristic function of the intervals K, D_σ and $(t^n, t^{n+1}]$ respectively.

For discrete functions q and v defined on the primal and dual mesh, respectively, we define a discrete $\mathrm{L}^1((0,T);\mathrm{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |q_L^n - q_K^n|, \qquad \|v\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\tilde{\mathcal{E}}_{\mathrm{int}}} |v_{\sigma'}^n - v_\sigma^n|,$$

and a discrete $\mathrm{L}^1(\Omega;\mathrm{BV}((0,T)))$ norm by:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K\in\mathcal{M}} |K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma\in\mathcal{E}} |D_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

For the consistency result that we are seeking (Theorem 2.4.2 below), we have to assume that a sequence of discrete solutions $\left(\rho^{(m)}, p^{(m)}, u^{(m)}\right)_{m\in\mathbb{N}}$ satisfies $\rho^{(m)} > 0$ and $p^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (2.20) is satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)^3$, *i.e.*:

$$0 < (\rho^{(m)})_K^n \le C,\ 0 < (p^{(m)})_K^n \le C, \quad \forall K \in \mathcal{M}^{(m)},\ \text{for } 0 \le n \le N^{(m)},\ \forall m \in \mathbb{N}, \tag{2.31}$$

and

$$|(u^{(m)})_\sigma^n| \le C, \quad \forall \sigma \in \mathcal{E}^{(m)},\ \text{for } 0 \le n \le N^{(m)},\ \forall m \in \mathbb{N}, \tag{2.32}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0 and u_0 belong to $\mathrm{L}^\infty(\Omega)$. We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{2.33}$$

We are not able to prove the estimates (2.31)–(2.33) for the solutions of the scheme; however, such inequalities are satisfied by the "interpolates" (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* with a limited sequence of meshes and time steps).

A weak solution to the continuous problem satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$- \int_0^T \int_\Omega \Big[\rho\, \partial_t\varphi + \rho\, u\, \partial_x \varphi\Big] \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\,\mathrm{d}x = 0, \tag{2.34a}$$

$$- \int_0^T \int_\Omega \Big[\rho\, u\, \partial_t\varphi + (\rho\, u^2 + p)\, \partial_x \varphi\Big] \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\,\mathrm{d}x = 0, \tag{2.34b}$$

$$p = \rho^\gamma. \tag{2.34c}$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks* (*i.e.* shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions). This is the result we are seeking and which we state in Theorem 2.4.2. In order to prove this theorem, we need some definitions of interpolates of regular test functions on the primal and dual mesh.

Definition 2.4.1 (Interpolates on one-dimensional meshes)**.** *Let Ω be an open bounded interval of $\mathbb{R}$, let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$, and let $\mathcal{M}$ be a mesh over Ω. The interpolate $\varphi_\mathcal{M}$ of φ on the primal mesh $\mathcal{M}$ is defined by:*

$$\varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \varphi_K^{n+1}\, \mathcal{X}_K\, \mathcal{X}_{[t^n,t^{n+1})}, \tag{2.35}$$

where, for $0 \le n \le N$ and $K \in \mathcal{M}$, we set $\varphi_K^n = \varphi(x_K, t^n)$, with x_K the mass center of K. The time discrete derivative of the discrete function $\varphi_\mathcal{M}$ is defined by:

$$\eth_t \varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t}\, \mathcal{X}_K\, \mathcal{X}_{[t^n,t^{n+1})}, \tag{2.36}$$

and its space discrete derivative by:

$$\eth_x \varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}\, \mathcal{X}_{D_\sigma}\, \mathcal{X}_{[t^n,t^{n+1})}. \tag{2.37}$$

Let $\varphi_\mathcal{E}$ be an interpolate of φ on the dual mesh, defined by:

$$\varphi_\mathcal{E} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} \varphi_\sigma^{n+1}\, \mathcal{X}_{D_\sigma}\, \mathcal{X}_{[t^n,t^{n+1})}, \tag{2.38}$$

where, for $1 \leq n \leq N$ and $\sigma \in \mathcal{E}$, we set $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$, with x_σ the abscissa of the interface σ. We also define the time and space discrete derivatives of this discrete function by:

$$\begin{aligned}
\eth_t \varphi_\mathcal{E} &= \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t}\, \mathcal{X}_{D_\sigma}\, \mathcal{X}_{[t^n,t^{n+1})}, \\
\eth_x \varphi_\mathcal{E} &= \sum_{n=0}^{N-1} \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \frac{\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}}{h_K}\, \mathcal{X}_K\, \mathcal{X}_{[t^n,t^{n+1})}.
\end{aligned} \tag{2.39}$$

We are now in position to state the following result.

Theorem 2.4.2 (Consistency of the one-dimensional explicit scheme)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (2.31)–(2.33) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^3$, for $1 \leq p < \infty$, to $(\bar\rho, \bar p, \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^3$.*

Then the limit $(\bar\rho, \bar p, \bar u)$ satisfies the system (2.34)*.*

Proof. It is clear that, with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The proof of this theorem is thus obtained by passing to the limit in the scheme for the mass balance equation first, and then for the momentum balance equation.

Mass balance equation – Let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$. Let $m \in \mathbb{N}$, $\mathcal{M}^{(m)}$ and $\delta t^{(m)}$ be given. Dropping for short the superscript $^{(m)}$, let $\varphi_\mathcal{M}$ be the interpolate of φ on the primal mesh and let $\eth_t\varphi_\mathcal{M}$ and $\eth_x\varphi_\mathcal{M}$ be its time and space discrete derivatives in the sense of Definition 2.4.1. Thanks to the regularity of φ, these functions respectively converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \geq 1$ (including $r = +\infty$), to φ, $\partial_t\varphi$ and $\partial_x\varphi$ respectively. In addition, $\varphi_\mathcal{M}(\cdot, 0)$ (which, for $K \in \mathcal{M}$ and $x \in K$, is equal to $\varphi_K^1 = \varphi(x, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \geq 1$. Since the support of φ is compact in $\Omega \times [0,T)$, for m large enough, the interpolate of φ vanishes at the boundary cells and at the last time step(s); hereafter, we systematically assume that we are in this case.

Let us multiply the first equation (2.9a) of the scheme by $\delta t\, \varphi_K^{n+1}$, and sum the result for $0 \leq n \leq N-1$ and $K \in \mathcal{M}$, to obtain $T_1^{(m)} + T_2^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} |K|(\rho_K^{n+1} - \rho_K^n)\, \varphi_K^{n+1}, \qquad T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (F_{\sigma'}^n - F_\sigma^n)\, \varphi_K^{n+1}.$$

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} |K|\, \rho_K^n\, \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{K\in\mathcal{M}} |K|\, \rho_K^0\, \varphi_K^1,$$

so that:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} \eth_t \varphi_{\mathcal{M}} \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\; \varphi_{\mathcal{M}}(x,0)\,\mathrm{d}x.$$

The boundedness of ρ_0 and the definition (2.26a) of the initial conditions for the scheme ensures that the sequence $((\rho^{(m)})^0)_{m\in\mathbb{N}}$ converges to ρ_0 in $\mathrm{L}^r(\Omega)$ for $r \geq 1$. Since, by assumption, the sequence of discrete solutions and of the interpolate time derivatives converge in $\mathrm{L}^r\big(\Omega \times [0,T)\big)$ for $r \geq 1$, we thus obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\,\mathrm{d}x.$$

Using the expression of the mass flux F_σ^n and reordering the sums in $T_2^{(m)}$, we get, remarking that $|D_\sigma| = h_\sigma$:

$$T_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} |D_\sigma|\; \rho_\sigma^n u_\sigma^n\, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}.$$

Since $|D_\sigma| = (|K| + |L|)/2$ and ρ_σ^n is the upwind approximation of ρ^n at the face σ, we can rewrite $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with

$$\mathcal{T}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \Big(\frac{|K|}{2}\rho_K^n + \frac{|L|}{2}\rho_L^n\Big) u_\sigma^n \, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

$$\mathcal{R}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} (\rho_K^n - \rho_L^n) \Big[\frac{|K|}{2}(u_\sigma^n)^- + \frac{|L|}{2}(u_\sigma^n)^+\Big] \, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

where, for $a \in \mathbb{R}$, $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$ (so $a = a^+ - a^-$). We have, for the term $\mathcal{T}_2^{(m)}$:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} u^{(m)} \eth_x \varphi_{\mathcal{M}} \, \mathrm{d}x \, \mathrm{d}t \quad \text{and} \quad \lim_{m\to+\infty} \mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \bar\rho \, \bar u \, \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

The remainder term $\mathcal{R}_2^{(m)}$ is bounded as follows, with $C_\varphi = \|\partial_x \varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}$:

$$\begin{aligned} |\mathcal{R}_2^{(m)}| &\leq C_\varphi \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}} |\rho_K^n - \rho_L^n| \, |D_\sigma| \, |u_\sigma^n| \\ &\leq C_\varphi \, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \, \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \, h^{(m)}, \end{aligned}$$

and therefore tends to zero when m tends to $+\infty$, by the assumed stability of the solution.

Momentum balance equation – Let $\varphi_{\mathcal{E}}$, $\eth_t \varphi_{\mathcal{E}}$ and $\eth_x \varphi_{\mathcal{E}}$ be the interpolate of φ on the dual mesh and its discrete time and space derivatives, in the sense of Definition 2.4.1, which converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \geq 1$ (including $r = +\infty$), to φ, $\partial_t \varphi$ and $\partial_x \varphi$ respectively. Let us multiply Equation (2.9c) by $\delta t \, \varphi_\sigma^{n+1}$, and sum the result for $0 \leq n \leq N-1$ and $\sigma \in \mathcal{E}_{\mathrm{int}}$. We obtain $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma| \, (\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) \, \varphi_\sigma^{n+1},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} \Big[F_L^n \, u_L^n - F_K^n \, u_K^n\Big] \, \varphi_\sigma^{n+1},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} (p_L^{n+1} - p_K^{n+1}) \, \varphi_\sigma^{n+1}.$$

Reordering the sums, we get for $T_1^{(m)}$:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\, \rho^n_{D_\sigma} u^n_\sigma\, \frac{\varphi^{n+1}_\sigma - \varphi^n_\sigma}{\delta t} - \sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\, \rho^0_{D_\sigma}\, u^0_\sigma\, \varphi^1_\sigma.$$

Thanks to the definition of the quantity ρ_{D_σ} (namely the fact that $|D_\sigma|\, \rho^n_{D_\sigma} = (|K|\, \rho^n_K + |L|\, \rho^n_L)/2$), we have:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, u^{(m)}\, \eth_t \varphi_{\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\; (u^{(m)})^0(x)\; \varphi_{\mathcal{E}}(x,0)\,\mathrm{d}x.$$

By the same arguments as for the mass balance equation, we therefore obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \bar u\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\,\mathrm{d}x.$$

Let us now turn to $T_2^{(m)}$. Reordering the sums and using the definition of the mass fluxes at the dual faces, we get:

$$\begin{aligned} T_2^{(m)} &= -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} F^n_K\, u^n_K\, (\varphi^{n+1}_{\sigma'} - \varphi^{n+1}_\sigma) \\ &= -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (\rho^n_\sigma u^n_\sigma + \rho^n_{\sigma'} u^n_{\sigma'})\, u^n_K\, (\varphi^{n+1}_{\sigma'} - \varphi^{n+1}_\sigma). \end{aligned}$$

Using the relation

$$\int_0^T \int_\Omega \rho^{(m)}\, {u^{(m)}}^2\, \eth_x \varphi_{\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \rho^n_K \left[(u^n_\sigma)^2 + (u^n_{\sigma'})^2\right] (\varphi^{n+1}_{\sigma'} - \varphi^{n+1}_\sigma),$$

we can rewrite the term $T_2^{(m)}$ as

$$T_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, {u^{(m)}}^2\, \eth_x \varphi_{\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t + \mathcal{R}_2^{(m)},$$

where:

$$\mathcal{R}_2^{(m)} = -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \Big[(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\, u_K^n - \rho_K^n \left((u_\sigma^n)^2 + (u_{\sigma'}^n)^2 \right) \Big]\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

Let us split this latter expression as $\mathcal{R}_2^{(m)} = \mathcal{R}_{21}^{(m)} + \mathcal{R}_{22}^{(m)}$, with:

$$\mathcal{R}_{21}^{(m)} = -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} u_\sigma^n \,(\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n)\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\mathcal{R}_{22}^{(m)} = -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} u_{\sigma'}^n \,(\rho_{\sigma'}^n u_K^n - \rho_K^n u_{\sigma'}^n)\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

Applying the identity $2(ab - cd) = (a-c)(b+d) + (a+c)(b-d),\ \forall (a,b,c,d) \in \mathbb{R}^4$, to the term $\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n$ and using the fact that the quantities $\rho_\sigma^n - \rho_K^n$ and $u_\sigma^n - u_K^n$ are either zero or differences of the density at two neighbouring cells and the velocity at two neighbouring faces respectively, we obtain for $\mathcal{R}_{21}^{(m)}$:

$$|\mathcal{R}_{21}^{(m)}| \leq C_\varphi \Big[\|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \Big]\, h^{(m)},$$

where the real number C_φ only depends on φ. Since the same estimate holds for $\mathcal{R}_{22}^{(m)}$, the remainder term $\mathcal{R}_2^{(m)}$ tends to zero when m tends to $+\infty$ and:

$$\lim_{m\to+\infty} T_2^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \bar u^2\, \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Let us finally study $T_3^{(m)}$. Reordering the sums, we obtain $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ with:

$$\mathcal{T}_3^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\mathcal{R}_3^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (p_K^{n+1} - p_K^n)\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

The remainder term reads:

$$\mathcal{R}_3^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n \left[(\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}) - (\varphi_{\sigma'}^n - \varphi_\sigma^n)\right] + \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^0\, (\varphi_{\sigma'}^1 - \varphi_\sigma^1),$$

and thus:

$$|\mathcal{R}_3^{(m)}| \leq |\Omega|\, C_\varphi \left[(T+1)\,\delta t^{(m)} + h^{(m)}\right] \|p\|_{\mathrm{L}^\infty(\Omega\times(0,T))},$$

where the real number C_φ only depends on (the first and second derivatives of) φ. Thus $\mathcal{R}_3^{(m)}$ tends to zero when m tends to $+\infty$ and, since

$$\mathcal{T}_3^{(m)} = -\int_0^T \int_\Omega p^{(m)}\, \eth_x \varphi_{\mathcal{M}} \,\mathrm{d}x\,\mathrm{d}t,$$

we obtain that:

$$\lim_{m\to+\infty} T_3^{(m)} = \int_0^T \int_\Omega \bar{p}\, \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Conclusion – Gathering the limits of all the terms of the mass and momentum balance equations concludes the proof. ☐

We now turn to the entropy condition (2.7). To this purpose, we need to introduce the following additional condition for a sequence of discretizations:

$$\lim_{m\to+\infty} \frac{\delta t^{(m)}}{\min_{K\in\mathcal{M}^{(m)}} h_K} = 0. \tag{2.40}$$

Note that this condition is slightly more restrictive that a standard CFL condition. It allows to bound the remainder term in the discrete elastic potential balance as stated in the following lemma.

Lemma 2.4.3. *Let Ω be an open bounded interval of $\mathbb{R}$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that the time step $\delta t^{(m)}$ tends to zero as $m \to \infty$, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (2.31)–(2.32). *In addition, we assume that $(\rho^{(m)})_{m\in\mathbb{N}}$ satisfies the following uniform BV estimate:*

$$\|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}, \tag{2.41}$$

and, for $\gamma < 2$ *only, is uniformly bounded by below,* i.e. *that there exists* $c > 0$ *such that:*

$$c \leq (\rho^{(m)})_K^n, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \ \forall m \in \mathbb{N}, \tag{2.42}$$

Let us suppose that the CFL condition (2.40) *hold. Let* $\mathcal{R}^{(m)}$ *be defined by:*

$$\mathcal{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} (R_K^{n+1})^-,$$

with R_K^{n+1} *given by* (2.25)*. Then:*

$$\lim_{m \to +\infty} \mathcal{R}^{(m)} = 0.$$

Proof. For $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$, with $\sigma = \overrightarrow{M|K}$ and $\sigma' = \overrightarrow{K|L}$, we write $R_K^{n+1} = (T_1)_K^{n+1} + (T_2)_K^{n+1} + (T_3)_K^{n+1}$, with:

$$\begin{aligned}
(T_1)_K^{n+1} &= \frac{1}{2} \frac{|K|}{\delta t} \, \mathcal{H}''(\overline{\rho}_{K,1}^n) \, (\rho_K^{n+1} - \rho_K^n)^2, \\
(T_2)_K^{n+1} &= \frac{1}{2} \Big[(u_{\sigma'}^n)^- \, \mathcal{H}''(\overline{\rho}_{\sigma'}^n) \, (\rho_K^n - \rho_L^n)^2 + (-u_\sigma^n)^- \, \mathcal{H}''(\overline{\rho}_\sigma^n) \, (\rho_K^n - \rho_M^n)^2 \Big], \\
(T_3)_K^{n+1} &= \Big[\rho_{\sigma'}^n \, u_{\sigma'}^n - \rho_\sigma^n \, u_\sigma^n \Big] \, \mathcal{H}''(\overline{\rho}_{K,2}^n) \, (\rho_K^{n+1} - \rho_K^n),
\end{aligned}$$

where $\overline{\rho}_{K,1}^n, \ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, $\overline{\rho}_{\sigma'}^n \in [\![\rho_K^n, \rho_L^n]\!]$ and $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_M^n]\!]$. The first two terms are non-negative, and thus $(R_K^{n+1})^- \leq |(T_3)_K^{n+1}|$. Since both ρ, u and, for $\gamma < 2$, $1/\rho$ are supposed to be bounded, there exists $C > 0$ such that:

$$\sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |(T_3)_K^{n+1}| \leq C \frac{\delta t^{(m)}}{\min_{K \in \mathcal{M}} h_K} \, \|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}},$$

which yields the conclusion by the assumption (2.40). ☐

Then we are now in position to state the following consistency result.

Theorem 2.4.4 (Entropy consistency, barotropic case)**.** *Let the assumptions of Theorem* 2.4.2 *hold. Let us suppose in addition that the considered sequence of discretization satisfies* (2.40)*, and that* $(\rho^{(m)})_{m \in \mathbb{N}}$ *satisfies the BV estimate* (2.41) *and, for* $\gamma < 2$*, the uniform control* (2.42) *of* $1/\rho^{(m)}$ *. Then the limit* $(\bar{\rho}, \, \bar{p}, \, \bar{u})$ *satisfies the entropy condition* (2.7)*.*

Proof. Let $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$, $\varphi \geq 0$. With the same notations for the interpolate of φ as in the preceding proof, we multiply the kinetic balance equation (2.21)-(2.22) by φ_σ^{n+1}, and the elastic potential balance (2.24)-(2.25) by φ_K^{n+1}, sum over the edges and cells respectively and over the time steps, to obtain the discrete version of (2.7):

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + \tilde{T}_1^{(m)} + \tilde{T}_2^{(m)} + \tilde{T}_3^{(m)} = -R^{(m)} - \tilde{R}^{(m)} \tag{2.43}$$

where:

$$\begin{aligned}
T_1^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\big]\ \varphi_K^{n+1},\\
T_2^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \big[\mathcal{H}(\rho_{\sigma'}^n)\, u_{\sigma'}^n - \mathcal{H}(\rho_\sigma^n)\, u_\sigma^n\big]\varphi_K^{n+1},\\
T_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \big[p_K^n(u_{\sigma'}^n - u_\sigma^n)\big]\ \varphi_K^{n+1},\\
\tilde{T}_1^{(m)} &= \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{|D_\sigma|}{\delta t}\big[\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n(u_\sigma^n)^2\big]\ \varphi_\sigma^{n+1},\\
\tilde{T}_2^{(m)} &= \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} \big[F_L^n\,(u_L^n)^2 - F_K^n\,(u_K^n)^2\big]\ \varphi_\sigma^{n+1},\\
\tilde{T}_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} (p_L^{n+1} - p_K^{n+1})\, u_\sigma^{n+1}\, \varphi_\sigma^{n+1},\\
R^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} R_K^{n+1}\, \varphi_K^{n+1}, \qquad \tilde{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} R_\sigma^{n+1}\, \varphi_\sigma^{n+1},
\end{aligned}$$

and the quantities R_K^{n+1} and R_σ^{n+1} are given by (the one-dimensional version of) Equation (2.25) and (2.22) respectively.

The fact that

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\bar{\rho})\ \partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \mathcal{H}(\rho_0)(x)\ \varphi(x,0)\,\mathrm{d}x,$$

is proven by the same technique as for passing to the limit in the term $T_1^{(m)}$ of the discrete mass balance equation in the proof Theorem 2.4.2, thanks to the fact that, with the assumed convergence of the sequence $(\rho^{(m)})_{m\in\mathbb{N}}$, the sequence $(\mathcal{H}(\rho^{(m)}))_{m\in\mathbb{N}}$ converge to $\mathcal{H}(\bar\rho)$ in $\mathrm{L}^r(\Omega\times(0,T))$, for $r\geq 1$. For $T_2^{(m)}$, we have, reordering the sums:

$$T_2^{(m)} = -\sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \mathcal{H}(\rho_\sigma^n)\, u_\sigma^n\, (\varphi_L^{n+1}-\varphi_K^{n+1}).$$

Let us write $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$, with

$$\begin{aligned} \mathcal{T}_2^{(m)} &= -\sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \big(|D_{K,\sigma}|\,\mathcal{H}(\rho_K^n)+|D_{L,\sigma}|\,\mathcal{H}(\rho_L^n)\big)\, u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma}, \\ \mathcal{R}_2^{(m)} &= \ -\sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \Big[|D_\sigma|\,\mathcal{H}(\rho_\sigma^n)-|D_{K,\sigma}|\,\mathcal{H}(\rho_K^n)-|D_{L,\sigma}|\,\mathcal{H}(\rho_L^n)\Big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma}. \end{aligned}$$

We have:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\rho^{(m)})\ u^{(m)}\ \eth_x\varphi_{\mathcal{M}}\,\mathrm{d}x\,\mathrm{d}t,$$

so

$$\lim_{m\to+\infty} T_2^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\bar\rho)\ \bar u\ \partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

The remainder term $\mathcal{R}_2^{(m)}$ satisfies:

$$|\mathcal{R}_2^{(m)}| \leq \sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} |\mathcal{H}(\rho_K^n)-\mathcal{H}(\rho_L^n)|\ u_\sigma^n\ |\varphi_L^{n+1}-\varphi_K^{n+1}|,$$

and so

$$|\mathcal{R}_2^{(m)}| \leq C_\varphi\ h^{(m)}\ \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\ \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}},$$

provided that a uniform (with respect to the faces, the time steps and the meshes) Lipschitz condition holds for $|\mathcal{H}(\rho_K^n)-\mathcal{H}(\rho_L^n)|$ which, in view of the expression of $\mathcal{H}$, requires that the sequence $(\rho^{(m)})_{m\in\mathbb{N}}$ be bounded by below away from zero when $\gamma=1$.

For the other terms at the left-hand side of (2.43), we refer to Chapter 3, Theorem 3.4.2. Finally, the remainder term $R^{(m)}$ is non-negative under the CFL condition

(2.23), while the positive part of $\tilde{R}^{(m)}$ tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$ under the assumption (2.40) by Lemma 2.4.3. The proof is thus complete. $\square$

2.5 Numerical results

We assess in this section the behaviour of the scheme on various test cases. For all these tests, we chose $p = \rho^2$ for the equation of state, so the solved system turns out to be the so-called shallow water equations. The exact solution of the Riemann problem is studied in Appendix A.

2.5.1 A first Riemann problem

We begin with a Riemann problem, *i.e.* a 1D problem which initial conditions consists in two constant states separated by a discontinuity. The chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = 5 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 10 \\ u_R = 7.5 \end{bmatrix}.$$

The computational domain is $\Omega = (0,1)$ and the final time is $T = 0.025$. The (known) analytical solution of this problem consists, from the left to the right, in a shock wave and a rarefaction wave, both travelling to the right, separated by constant states.

2.5.1.1 Results

The density and velocity obtained at $t = 0.025 = T$ with $h = 0.001$ and $\delta t = h/12$ are shown of Figures 2.1 and 2.2 respectively. In addition, we performed a convergence study, successively dividing by two the space and time steps (so keeping the CFL number constant). The difference between the computed and analytical solution at $t = 0.025$, measured in $\mathrm{L}^1(\Omega)$ norm, are reported in the following table:

space step	$h_0 = 1/250$	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
$\|\rho - \bar{\rho}\|_{\mathrm{L}^1(\Omega)}$	0.0449	0.0256	0.0135	0.00775	0.00429
$\|u - \bar{u}\|_{\mathrm{L}^1(\Omega)}$	0.0411	0.0233	0.0119	0.00696	0.00384

We observe an approximatively first-order convergence rate.

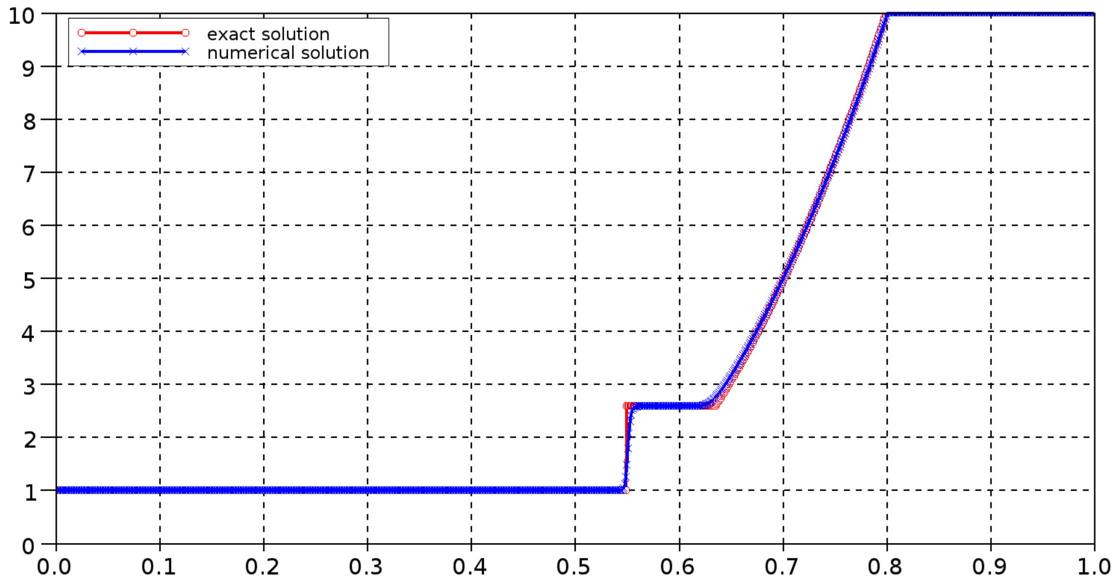


Figure 2.1: Test 1 – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.025$.

To complete the study, we performed a computation of the same problem, but subtracting a constant real number to the left and right velocity, in such a way that the velocity on the intermediate state nearly vanishes. In this case, we observe spurious oscillations on the solution (see Figures 2.3 and 2.4), probably due to the fact that the numerical diffusion in the scheme vanishes. However, adding an artificial viscosity term in the discrete momentum balance equation, with a constant viscosity equal to $0.5\,\rho\,h$ (so equal to the upwind viscosity which would be associated to a velocity equal to 1) completely cures the problem (see Figures 2.5 and 2.6). This observation strongly supports the idea to build a higher order scheme using an *a posteriori* fitted viscosity technique, as in the so-called entropy viscosity method [21, 22]; this work is underway.

When we substract once again a constant to the velocity at both left and right state, and so the velocity at the intermediate becomes negative, we recover an oscillation-free solution without adding any viscosity (Figures 2.7 and 2.8).

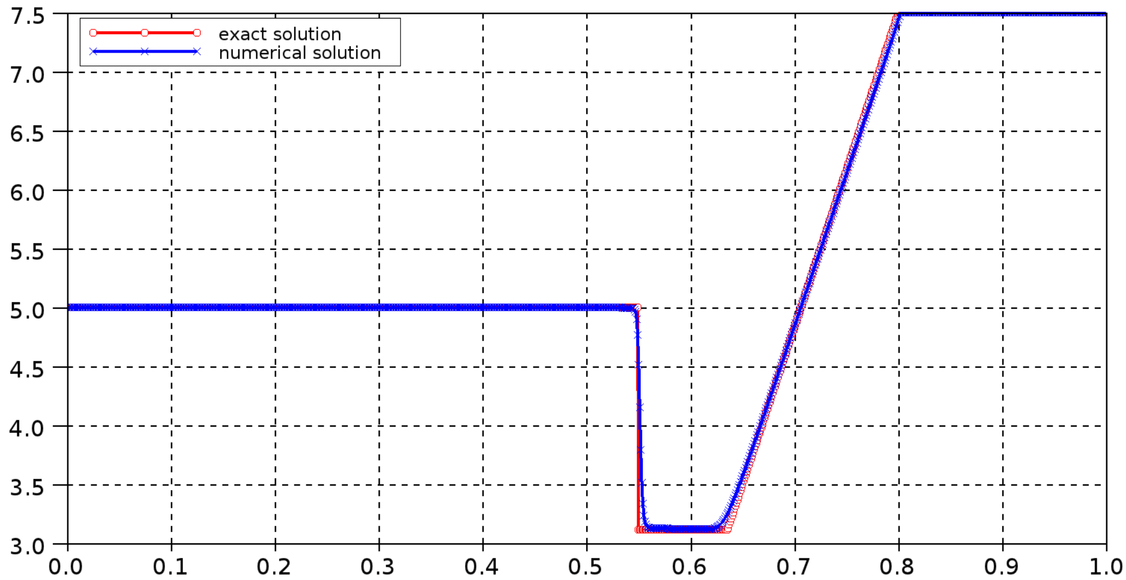


Figure 2.2: Test 1 – $h = 0.001,\ \delta t = h/12$ –Velocity at $t = 0.025$.

2.5.1.2 On a naive scheme

We also test the "naive" explicit scheme obtained by evaluating all the terms, except of course the time-derivative one, at time t^n. In the one dimensional setting and with the same notations as in Section 2.4, this scheme thus reads:

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{2.44a}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^n - p_K^n = 0, \tag{2.44b}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{2.44c}$$

Hereafter and on the figure captions, this scheme is referred to by the $\rho \rightsquigarrow u \rightsquigarrow p$ scheme (since the pressure is updated after the computation of the velocity rather than after the computation of the density).

The computed density and velocity at time $T = 0.025$ are plotted on figures 2.9

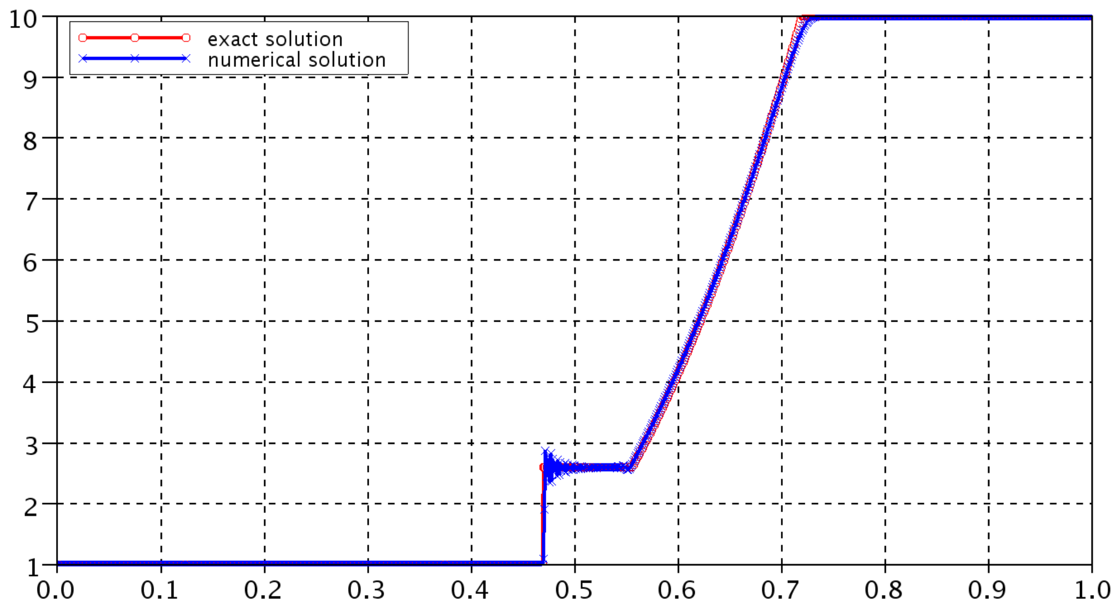


Figure 2.3: Test 1 modified to obtain a nearly vanishing velocity at the intermediate state, viscosity $= 0$ – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.025$.

and 2.10 respectively. From these results, it appears clearly that the $\rho \rightsquigarrow u \rightsquigarrow p$ scheme generates discontinuities in the rarefaction wave, and further experiments show that this phenomenon is not cured by a decrease of the time and space step; this seems to be connected to the fact that, for this variant, we cannot prove that the limits of converging sequences satisfy the entropy condition (in fact, they probably do not). When trying to do so, in our proof and from a purely technical point of view, the trouble comes from the fact that the pressure gradient term which appears in the kinetic energy balance reads $\boldsymbol{u}^{n+1}\boldsymbol{\nabla}p^n$ and it seems difficult to make the counterpart (*i.e.* $p^n\mathrm{div}(\boldsymbol{u}^{n+1})$), with the corresponding time levels, appear in the elastic potential balance, starting from a mass balance with a convection term written with $\boldsymbol{u}^n$; hence a discretization of the momentum balance equation with an updated pressure gradient term $\boldsymbol{\nabla}p^{n+1}$, and thus the inversion of steps in the algorithm, to get the "reference variant" proposed in this chapter.

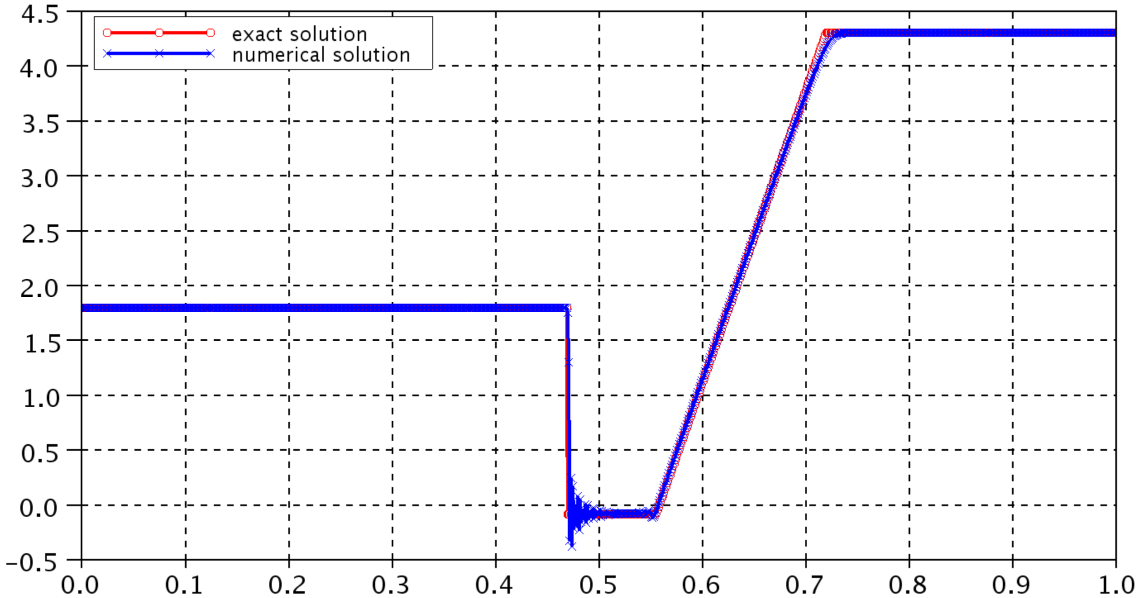


Figure 2.4: Test 1 modified to obtain a nearly vanishing velocity at the intermediate state, viscosity $= 0$ – $h = 0.001,\ \delta t = h/12$ – Velocity at $t = 0.025$.

2.5.2 Problems involving vacuum zones in the flow

The objective of the two tests presented in this section is to check that the time step does not have to be drastically reduced in the presence of vacuum. Both are Riemann problems, posed on $\Omega = (0,1)$.

We first begin with a case where the vacuum is initially present, at the right initial state:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = 1 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 0 \\ u_R = 0 \end{bmatrix}.$$

In the computer code, ρ_R is fixed as $\rho_R = 10^{-20}$, to prevent divisions by zero due to imprudent programming. The results obtained at $t = 0.05$ are plotted on Figure 2.11 (density) and Figure 2.12 (velocity); they have been obtained with $h = 0.001$ and a constant time step equal to $\delta t = h/8$, which seems to be near to the stability limit. We observe that the prediction velocity is rather poor near to the vacuum front; we however check on Figure 2.13 that the scheme converges to the right solution; moreover, Figure

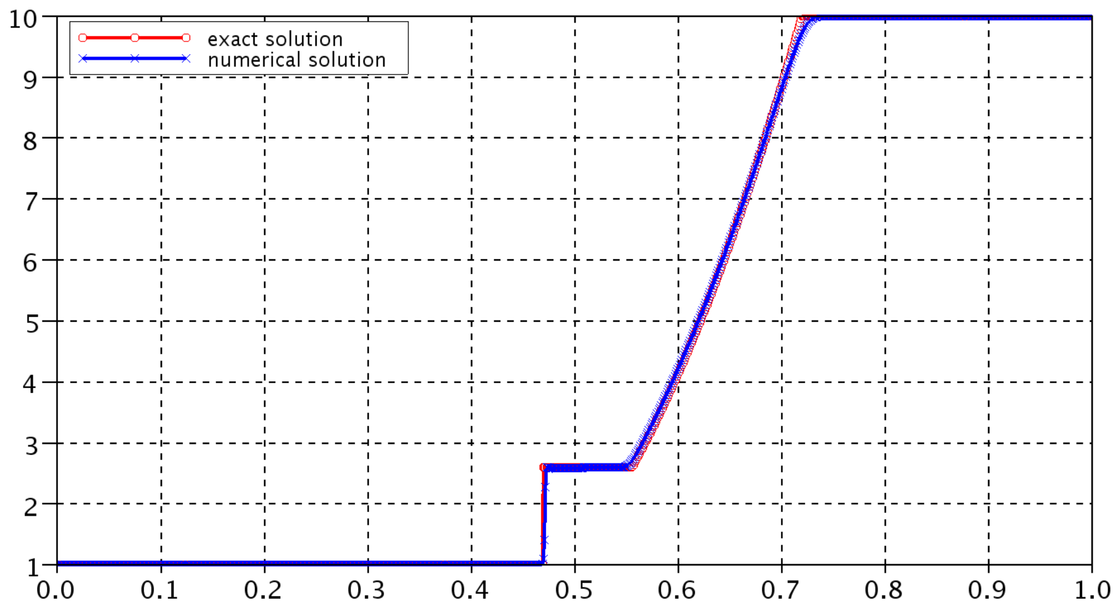


Figure 2.5: Test 1 modified to obtain a nearly vanishing velocity at the intermediate state, viscosity $= 0.5$ – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.025$.

2.14 shows that the quantity $\rho\, u$ (which is, in this case, the quantity of physical interest) is in fact obtained with a reasonable accuracy with the coarsest meshes of this study.

We now turn a case where the chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = -8 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 1 \\ u_R = 8 \end{bmatrix}.$$

In this case, the solution consists in an intermediate state corresponding to vacuum connected to the left and right initial states by rarefaction waves. Computed density and velocity at $t = 0.03$, with $h = 0.001$ and $\delta t = h/12$, are plotted on Figures 2.15 and 2.16 respectively. Once again, the behaviour of the scheme is satisfactory.

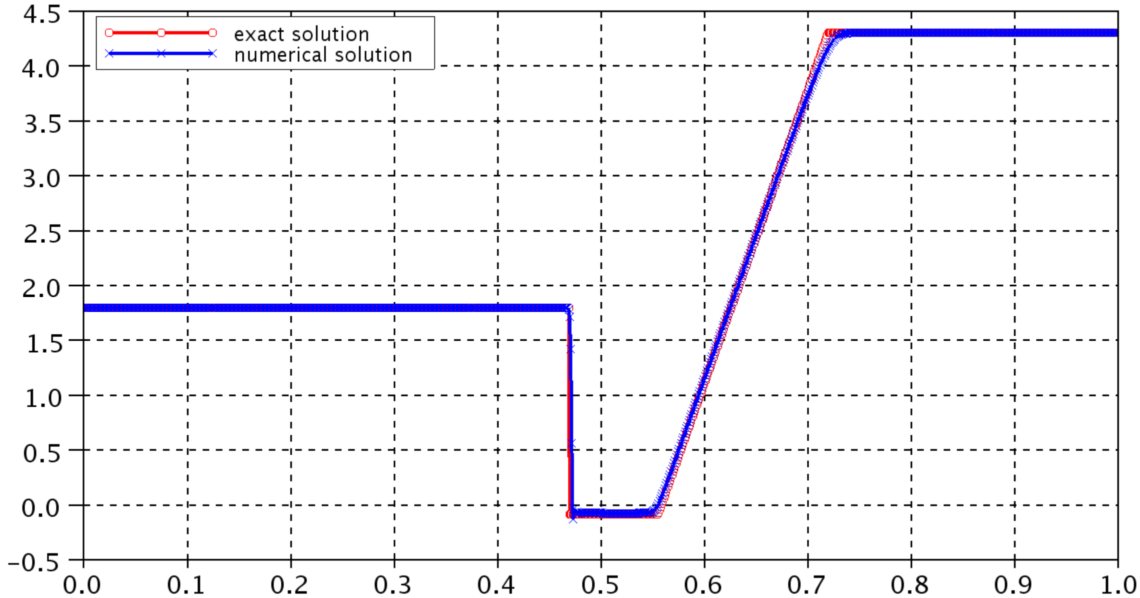


Figure 2.6: Test 1 modified to obtain a nearly vanishing velocity at the intermediate state, viscosity $= 0.5$ – $h = 0.001,\ \delta t = h/12$ – Velocity at $t = 0.025$.

2.6 Conclusion

We have presented in this chapter an explicit scheme based on staggered meshes for the hyperbolic system of the barotropic Euler equations. This algorithm uses a very simple first-order upwinding strategy which consists, equation by equation, to implement an upwind discretization with respect of the material velocity of the convection term. Under CFL-like conditions based on the material velocity only (by opposition to the celerity of waves), this scheme preserves the positivity of the density and the pressure, and has been shown to be consistent for 1D problems, in the sense that, if a sequence of numerical solutions obtained with more and more refined meshes (and, accordingly, smaller and smaller time steps) converges, then the limit is a weak entropy solution to the continuous problem. This theoretical result may probably be extended to the multi-dimensional case, and this work is now being undertaken. The proposed scheme has a natural extension to the full Euler equations, which is the topic of next chapter. Note also that a partial time-implicitation, using pressure correction techniques, has been shown to yield consistent

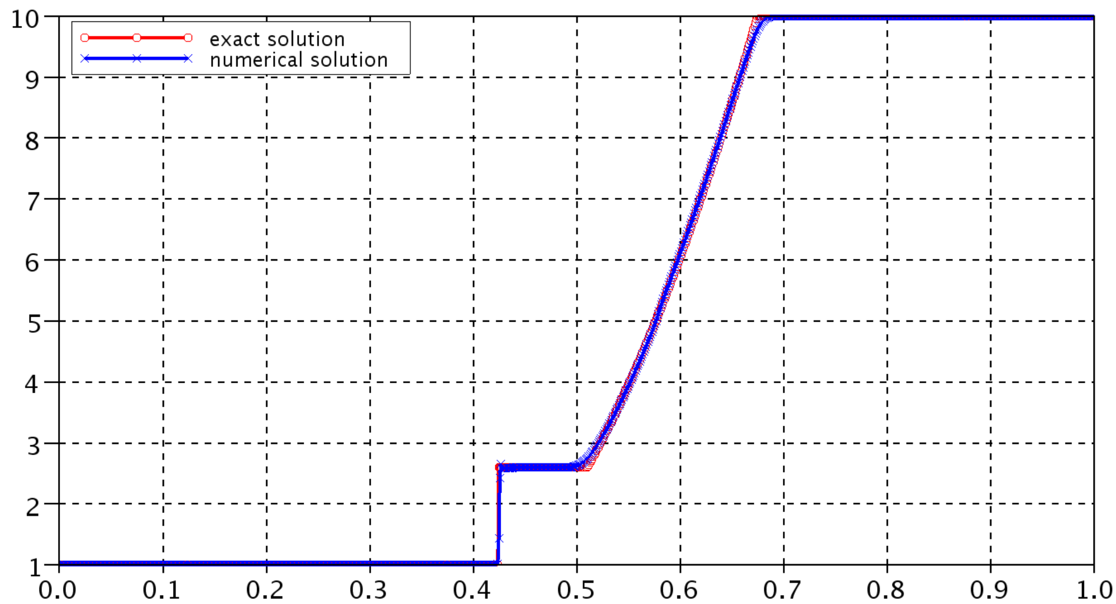


Figure 2.7: Test 1 modified to obtain a negative velocity at the intermediate state – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.025$.

unconditionally stable schemes [26, 27].

Numerical studies show that the proposed algorithm is stable, even if the largest time step before blow-up is smaller than suggested by the above-mentioned CFL conditions. This behaviour had to be expected, since these CFL conditions only involve the velocity (and not the celerity of the acoustic waves): indeed, were they the only limitation, we would have obtained an explicit scheme stable up to the incompressible limit. However, the mechanisms leading to the blow-up of the scheme (or, conversely, the way to fix the time step to ensure stability) remain to be understood.

In addition, numerical experiments show that some oscillations appear near stagnation points, where the numerical diffusion brought by the upwinding vanishes. These oscillations are damped by a small amount of artificial (physical-like) viscosity, and this suggests to implement techniques consisting in adding to the scheme such a diffusion term, with a viscosity monitored by an *a posteriori* (*i.e.* performed in view of the results of the previous time step) analysis of the solution, as the so-called entropy-viscosity technique. Besides, such an extension should allow to design a more accurate scheme, based

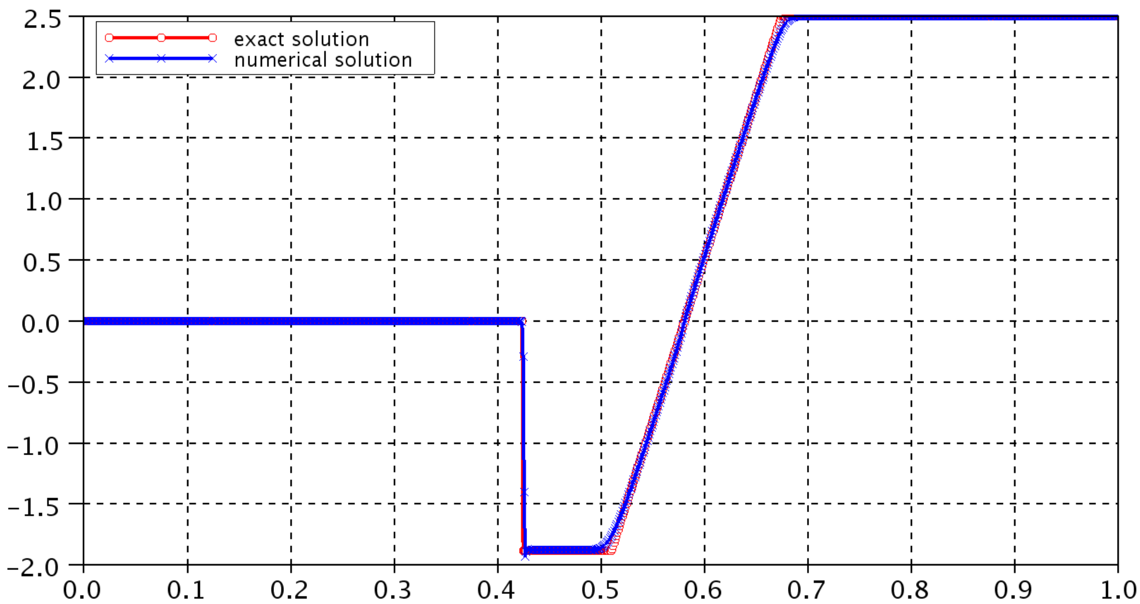


Figure 2.8: Test 1 modified to obtain a negative velocity at the intermediate state – $h = 0.001,\ \delta t = h/12$ – Velocity at $t = 0.025$.

on higher-order numerical fluxes. This work is underway.

Last but not least, since the proposed scheme uses very simple numerical fluxes, it is well suited to large multi-dimensional parallel computing applications. This is the topic of ongoing studies at IRSN.

2.7 Appendix

2.7.1 Some results concerning explicit finite volumes convection operators

We begin with the convection operator appearing in the mass balance equation, which reads, in the continuous problem, $\rho \to \mathcal{C}(\rho) = \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u})$, where $\boldsymbol{u}$ stands for a given velocity field, which is not assumed to satisfy any divergence constraint. Let ψ be

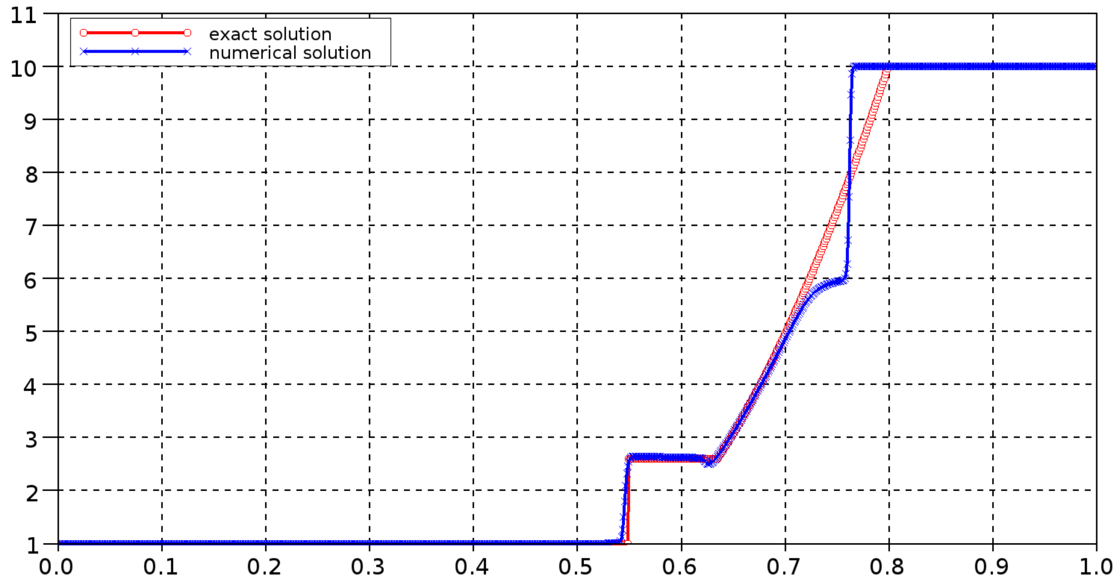


Figure 2.9: Test 1, $\rho \rightsquigarrow u \rightsquigarrow p$ scheme – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.025$.

a regular function from $(0, +\infty)$ to $\mathbb{R}$; then:

$$\begin{aligned}\psi'(\rho)\,\mathcal{C}(\rho) &= \psi'(\rho)\,\partial_t(\rho) + \psi'(\rho)\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho + \psi'(\rho)\,\rho\,\mathrm{div}\boldsymbol{u} \\ &= \partial_t(\psi(\rho)) + \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi(\rho) + \rho\,\psi'(\rho)\,\mathrm{div}\boldsymbol{u},\end{aligned}$$

so adding and subtracting $\psi(\rho)\,\mathrm{div}\boldsymbol{u}$ yields:

$$\psi'(\rho)\,\mathcal{C}(\rho) = \partial_t\big(\psi(\rho)\big) + \mathrm{div}\big(\psi(\rho)\boldsymbol{u}\big) + \big(\rho\psi'(\rho) - \psi(\rho)\big)\,\mathrm{div}\boldsymbol{u}. \tag{2.45}$$

This computation is of course completely formal and only valid for regular functions ρ and $\boldsymbol{u}$. The following lemma states a discrete analogue to (2.45).

Lemma 2.7.1. *Let P be a polygonal (resp. polyhedral) bounded set of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$), and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let ψ be a twice continuously differentiable function defined over $(0, +\infty)$. Let $\rho_P^* > 0$, $\rho_P > 0$, $\delta t > 0$; consider three families $(\rho_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}_+ \setminus \{0\}$, $(V_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ and $(F_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ such that*

$$\forall \eta \in \mathcal{E}(P), \qquad F_\eta^* = \rho_\eta^*\, V_\eta^*,$$

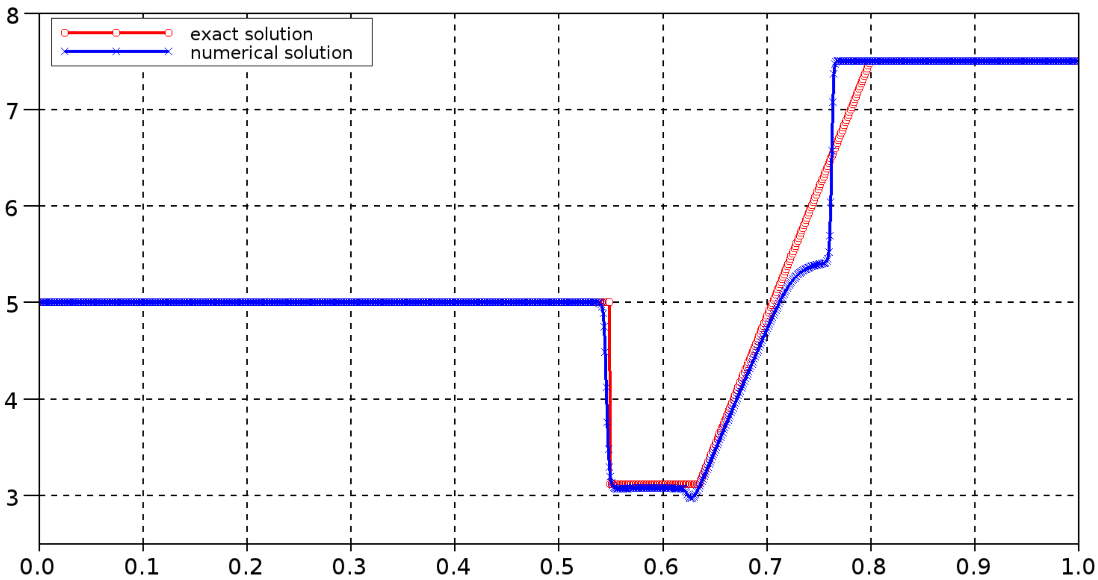


Figure 2.10: Test 1, $\rho \rightsquigarrow u \rightsquigarrow p$ scheme – $h = 0.001,\ \delta t = h/12$ – Velocity at $t = 0.025$.

Let $R_{P,\delta t}$ be defined by:

$$R_{P,\delta t} = \Big[\frac{|P|}{\delta t}\,(\rho_P - \rho_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta^*\Big]\,\psi'(\rho_P) \\ - \frac{|P|}{\delta t}\,[\psi(\rho_P) - \psi(\rho_P^*)] + \sum_{\eta \in \mathcal{E}(P)} \psi(\rho_\eta^*) V_\eta^* + [\rho_P^* \psi'(\rho_P^*) - \psi(\rho_P^*)] \sum_{\eta \in \mathcal{E}(P)} V_\eta^*.$$

Then this quantity may be expressed as follows:

$$R_{P,\delta t} = \frac{1}{2}\frac{|P|}{\delta t}\,(\rho_P - \rho_P^*)^2\,\psi''(\overline{\rho}_P^{(1)}) + \sum_{\eta \in \mathcal{E}(P)} V_\eta^* \rho_\eta^*\,(\rho_P - \rho_P^*)\,\psi''(\overline{\rho}_P^{(2)}) \\ - \frac{1}{2} \sum_{\eta \in \mathcal{E}(P)} V_\eta^*\,(\rho_P^* - \rho_\eta^*)^2\,\psi''(\overline{\rho}_\eta^*),$$

where $\overline{\rho}_P^{(1)},\ \overline{\rho}_P^{(2)} \in [\![\rho_P, \rho_P^*]\!]$ and $\forall \eta \in \mathcal{E}(P)$, $\overline{\rho}_\eta^* \in [\![\rho_P^*, \rho_\eta^*]\!]$. We recall that, for $a,\ b \in \mathbb{R}$, we denote by $[\![a, b]\!]$ the interval $[\![a, b]\!] = \{\theta a + (1-\theta) b,\ \theta \in [0, 1]\}$.

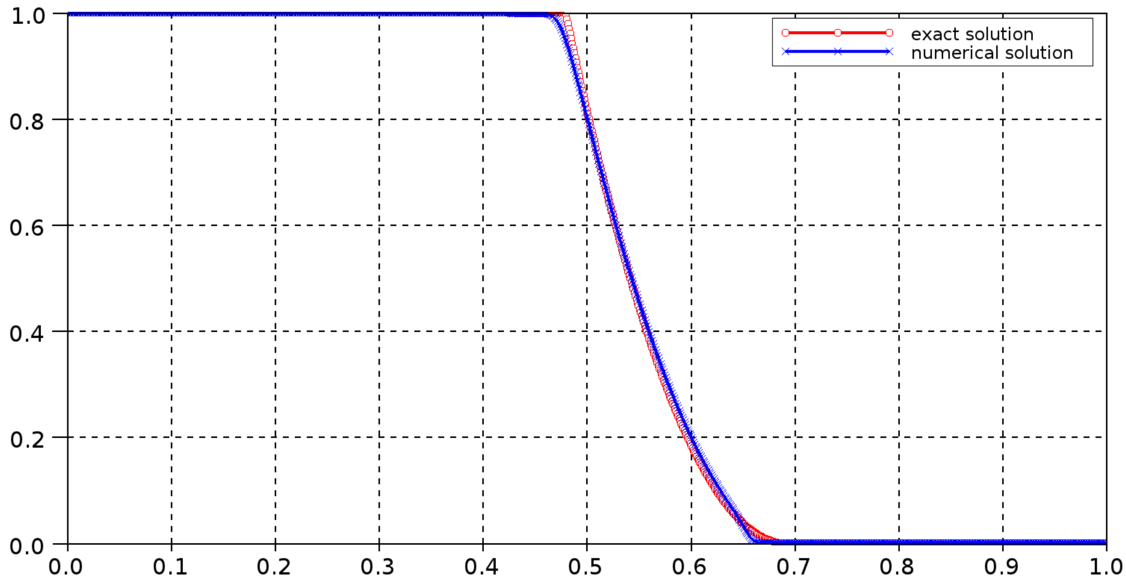


Figure 2.11: Riemann problem with vacuum at the right state – $h = 0.001,\ \delta t = h/8$ – Density at $t = 0.05$.

Proof. By the definition of F^*_η, we have:

$$\Big[\frac{|P|}{\delta t}(\rho_P - \rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} F^*_\eta\Big]\ \psi'(\rho_P)$$
$$= \frac{|P|}{\delta t}(\rho_P - \rho^*_P)\,\psi'(\rho_P) + \sum_{\eta\in\mathcal{E}(P)} \rho^*_\eta V^*_\eta \psi'(\rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} \rho^*_\eta V^*_\eta \big[\psi'(\rho_P) - \psi'(\rho^*_P)\big]. \tag{2.46}$$

By Taylor expansions of ψ, there exists two real numbers $\overline{\rho}^{(1)}_P$ and $\overline{\rho}^{(2)}_P \in [\![\rho^*_P, \rho_P]\!]$ and a family of real numbers $(\overline{\rho}^*_\eta)_{\eta\in\mathcal{E}(P)}$ satisfying, $\forall \eta \in \mathcal{E}(P)$, $\overline{\rho}^*_\eta \in [\![\rho^*_P, \rho^*_\eta]\!]$, and such that:

$$(\rho_P - \rho^*_P)\,\psi'(\rho_P) = \psi(\rho_P) - \psi(\rho^*_P) + \frac{1}{2}\,(\rho_P - \rho^*_P)^2\,\psi''(\overline{\rho}^{(1)}_P),$$
$$\rho^*_\eta \psi'(\rho^*_P) = \psi(\rho^*_\eta) + [\rho^*_P \psi'(\rho^*_P) - \psi(\rho^*_P)] - \frac{1}{2}\,(\rho^*_\eta - \rho^*_P)^2\,\psi''(\overline{\rho}^*_\eta),$$
$$\psi'(\rho_P) - \psi'(\rho^*_P) = (\rho_P - \rho^*_P)\,\psi''(\overline{\rho}^{(2)}_P).$$

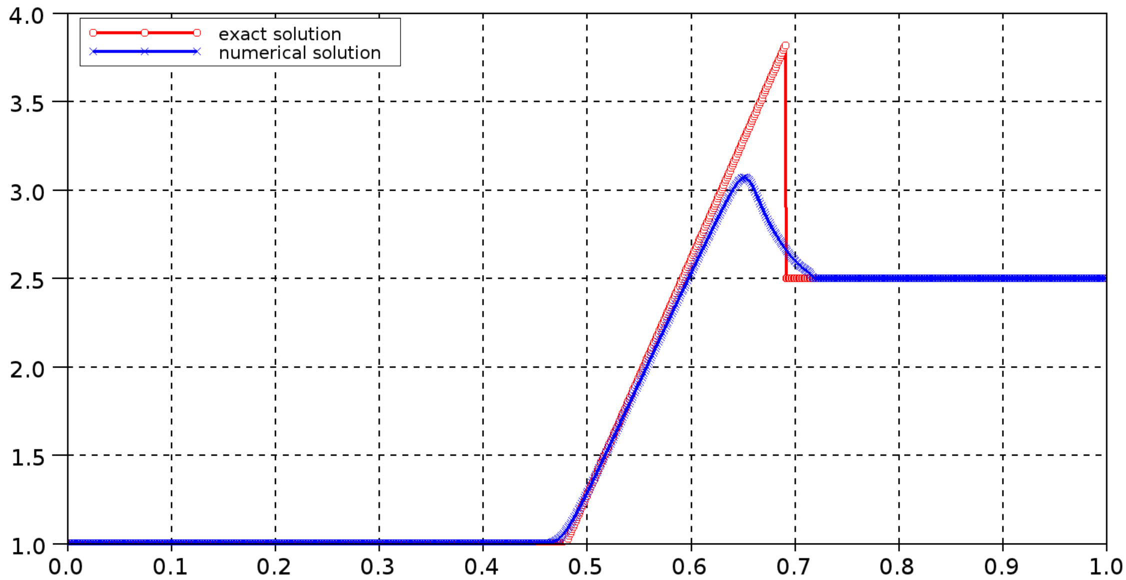


Figure 2.12: Riemann problem with vacuum at the right state – $h = 0.001,\ \delta t = h/8$ – Velocity at $t = 0.05$.

Substituting in (2.46) yields the result we are seeking. $\square$

We now turn to the convection operator appearing in the momentum balance equation, which reads, in the continuous setting, $z \to \mathcal{C}_\rho(z) = \partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u})$, where ρ (resp. $\boldsymbol{u}$) stands for a given scalar (resp. vector) field; we wish to obtain some property of C_ρ under the assumption that ρ and $\boldsymbol{u}$ satisfy a mass balance equation, *i.e.* $\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0$. Formally, using twice the assumption $\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0$ yields:

$$\psi'(z)\, \mathcal{C}_\rho(z) = \psi'(z)\big[\partial_t(\rho\, z) + \mathrm{div}(\rho\, z\, \boldsymbol{u})\big] = \psi'(z)\rho\big[\partial_t z + \boldsymbol{u}\cdot\boldsymbol{\nabla} z\big]$$
$$= \rho\big[\partial_t\psi(z) + \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi(z)\big] = \partial_t\big(\rho\,\psi(z)\big) + \mathrm{div}\big(\rho\,\psi(z)\,\boldsymbol{u}\big).$$

Taking for z a component of the velocity field, this relation is the central argument used to derive the kinetic energy balance. The following lemma states a discrete counterpart of this identity, for a finite volume first-order explicit convection operator.

Lemma 2.7.2. *Let P be a polygonal (resp. polyhedral) bounded set of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let $\rho_P^* > 0$, $\rho_P > 0$, $\delta t > 0$, and*

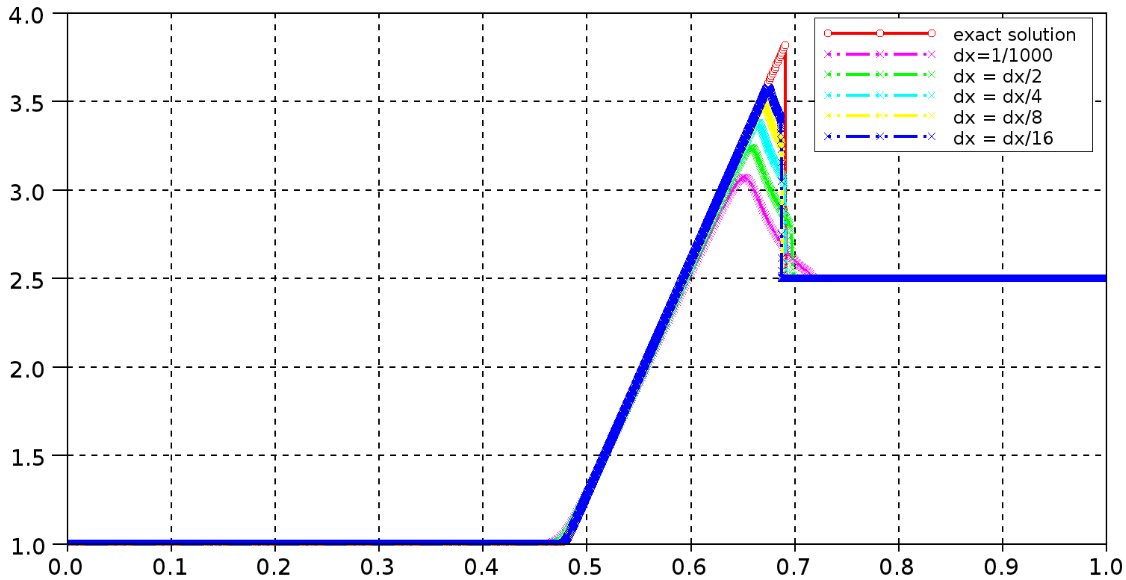


Figure 2.13: Riemann problem with vacuum at the right state – $h = h_0 = 0.001$ to $h = h_0/16$, $\delta t = h/8$ – Velocity at $t = 0.05$.

$(F_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ *be such that*

$$\frac{|P|}{\delta t}\,(\rho_P - \rho_P^*) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^* = 0. \tag{2.47}$$

Let ψ *be a twice continuously differentiable function defined over* $(0,+\infty)$. *For* $u_P^* \in \mathbb{R}$, $u_P \in \mathbb{R}$ *and* $(u_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ *let us define:*

$$\begin{aligned} R_{P,\delta t} = \Big[\frac{|P|}{\delta t}\,\big(\rho_P\, u_P - \rho_P^*\, u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\, u_\eta^*\Big]\,\psi'(u_P) \\ - \Big[\frac{|P|}{\delta t}\,\big[\rho_P\,\psi(u_P) - \rho_P^*\,\psi(u_P^*)\big] + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,\psi(u_\eta^*)\Big]. \end{aligned}$$

Then:

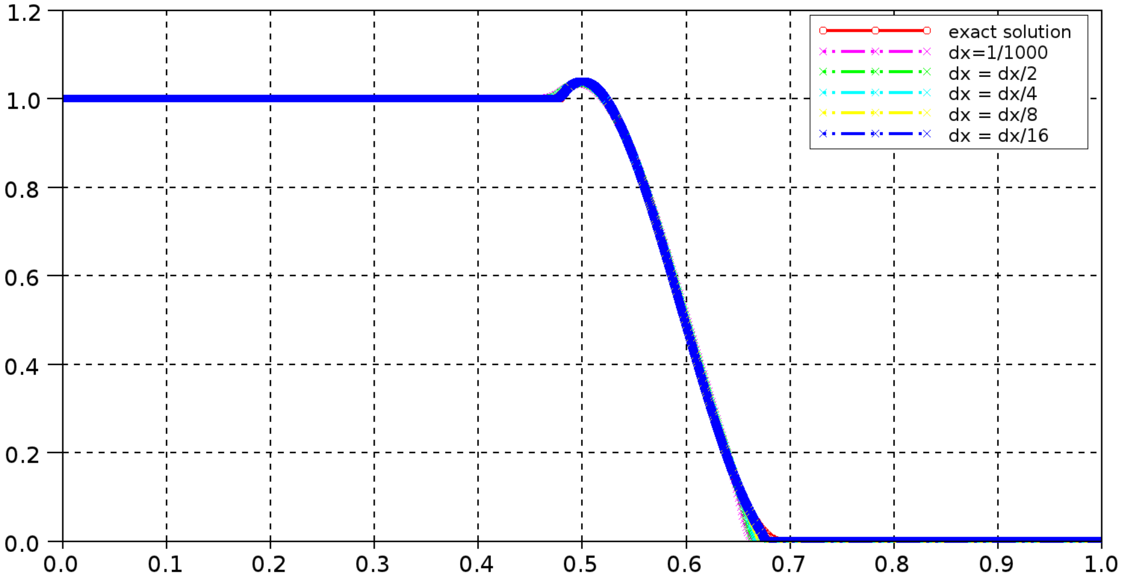


Figure 2.14: Riemann problem with vacuum at the right state – $h = h_0 = 0.001$ to $h = h_0/16,\ \delta t = h/8$ – Mass flowrate at $t = 0.05$.

(i) *the remainder term $R_{P,\delta t}$ reads:*

$$R_{P,\delta t} = \frac{1}{2}\,\frac{|P|}{\delta t}\,\rho_P\,(u_P - u_P^*)^2\psi''(\overline{u}_P^{(1)}) - \frac{1}{2}\sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_p^*)^2\psi''(\overline{u}_\eta^*)$$
$$+ \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_P^*)\,(u_P - u_P^*)\,\psi''(\overline{u}_P^{(2)}) \tag{2.48}$$

with $\overline{u}_P^{(1)}, \overline{u}_P^{(2)} \in [\![u_P, u_P^*]\!]$*, and* $\forall \eta \in \mathcal{E}(P)$*,* $\overline{u}_\eta^* \in [\![u_\eta^*, u_P^*]\!]$.

(ii) *If we suppose that the function ψ is convex and that $u_\eta^* = u_P^*$ as soon as $F_\eta^* \geq 0$, then $R_{P,\delta t}$ is non-negative under the CFL condition:*

$$\delta t \leq \frac{|P|\,\rho_P\,\underline{\psi}''_P}{\sum_{\eta\in\mathcal{E}(P)}(F_\eta^*)^-\,(\overline{\psi}''_P)^2/\underline{\psi}''_\eta}, \tag{2.49}$$

where $\underline{\psi}''_P = \min_{s\in[\![u_P,u_P^*]\!]}\psi''(s)$, $\overline{\psi}''_P = \max_{s\in[\![u_P,u_P^*]\!]}\psi''(s)$ *and* $\underline{\psi}''_\eta = \min_{s\in[\![u_P^*,u_\eta^*]\!]}\psi''(s)$.

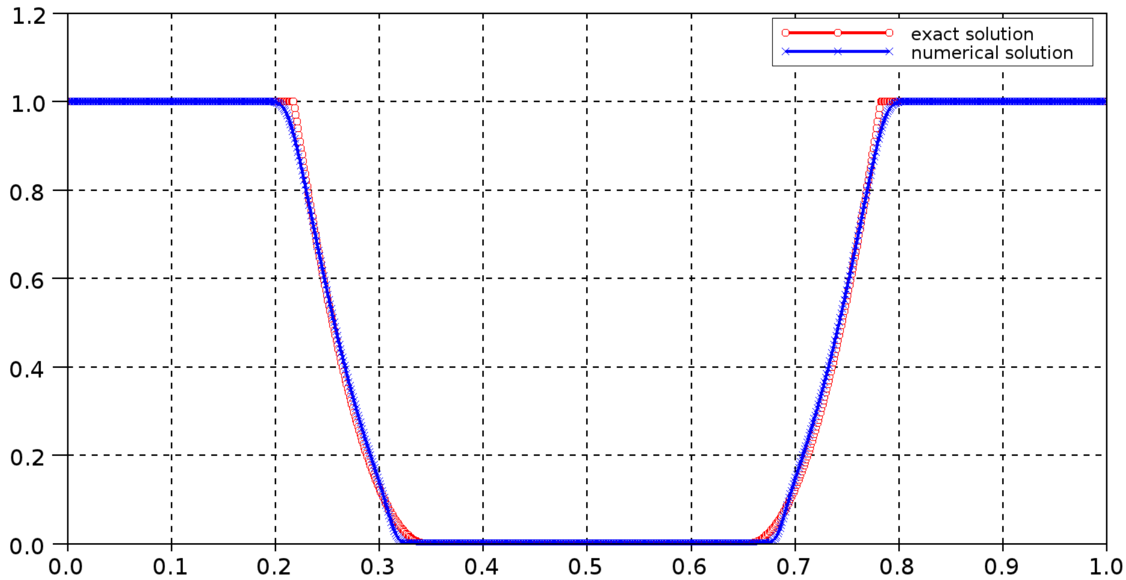


Figure 2.15: Riemann problem with vacuum appearance – $h = 0.001,\ \delta t = h/12$ – Density at $t = 0.03$.

For $\psi(s) = s^2/2$ (and therefore $\psi''(s) = 1,\ \forall s \in (0, +\infty)$), this CFL condition simply reads:

$$\delta t \leq \frac{|P|\,\rho_P}{\sum_{\eta\in\mathcal{E}(P)} (F_\eta^*)^-}. \tag{2.50}$$

Proof. Let T_P be defined by:

$$T_P = \Big[\frac{|P|}{\delta t}\,\big(\rho_P\, u_P - \rho_P^*\, u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\ u_\eta^*\Big]\ \psi'(u_P).$$

Using equation (2.47) multiplied by u_P^*, we obtain:

$$T_P = \Big[\frac{|P|}{\delta t}\,\rho_P\,\big(u_P - u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\ (u_\eta^* - u_P^*)\Big]\ \psi'(u_P).$$

We now define the remainder terms r_P and $(r_\eta^*)_{\eta\in\mathcal{E}(P)}$ by:

$$r_P = (u_P - u_P^*)\ \psi'(u_P) - \big[\psi(u_P) - \psi(u_P^*)\big],\quad r_\eta^* = (u_P^* - u_\eta^*)\ \psi'(u_P^*) - \big[\psi(u_P^*) - \psi(u_\eta^*)\big].$$

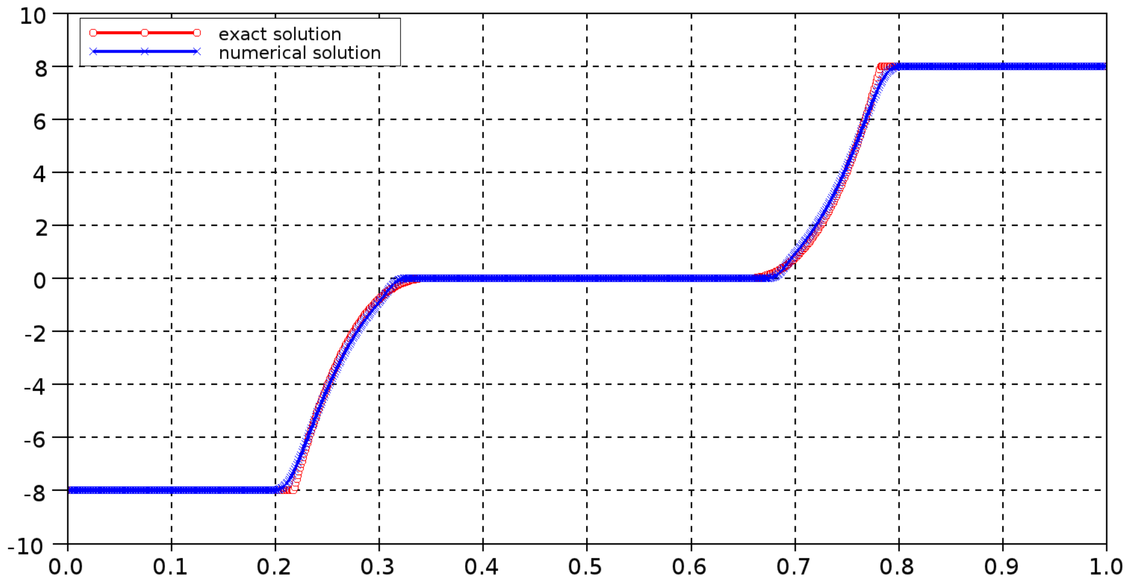


Figure 2.16: Riemann problem with vacuum appearance – $h = 0.001,\ \delta t = h/12$ – Mass flowrate at $t = 0.03$.

With these notations, we get:

$$T_P = \frac{|P|}{\delta t}\,\rho_P\,\big[\psi(u_P) - \psi(u_P^*)\big] + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\big[\psi(u_\eta^*) - \psi(u_P^*)\big] \\ + \frac{|P|}{\delta t}\,\rho_P\,r_P - \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,r_\eta^* + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_P^*)\,\big(\psi'(u_P) - \psi'(u_P^*)\big).$$

Using once again equation (2.47), this time multiplied by $\psi(u_P^*)$, we obtain:

$$T_P = \frac{|P|}{\delta t}\,\big[\rho_P\psi(u_P) - \rho_P^*\psi(u_P^*)\big] + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\psi(u_\eta^*) \\ + \frac{|P|}{\delta t}\,\rho_P\,r_P - \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,r_\eta^* + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_P^*)\,\big(\psi'(u_P) - \psi'(u_P^*)\big).$$

The expression (2.48) of the remainder term $R_{P,\delta t}$ follow by remarking that, by a Taylor expansion, there exists $\overline{u}_P^{(1)}, \overline{u}_P^{(2)} \in [\![u_P, u_P^*]\!]$, and $\forall \eta \in \mathcal{E}(P)$, $\overline{u}_\eta^* \in [\![u_\eta^*, u_P^*]\!]$ such that:

$$r_P = \frac{1}{2}\, \psi''(\overline{u}_P^{(1)})\, (u_P - u_P^*)^2, \qquad r_\eta^* = \frac{1}{2}\, \psi''(\overline{u}_\eta^*)\, (u_\eta^* - u_P^*)^2,$$
$$\psi'(u_P) - \psi'(u_P^*) = \psi''(\overline{u}_P^{(2)})\, (u_P - u_P^*).$$

If ψ is convex, r_P is non-negative. If, in addition, $u_P^* - u_\eta^*$ vanishes $\forall \eta \in \mathcal{E}(P)$ when F_η^* is non-negative, $-r_\eta^*$ is non-negative. By Young's inequality, the last term in $R_{P,\delta t}$ may be bounded as follows:

$$\begin{aligned} &\Big| \sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\, (u_\eta^* - u_P^*)\, (u_P - u_P^*)\, \psi''(\overline{u}_P^{(2)}) \Big| \\ &\leq \frac{\psi''(\overline{u}_P^{(2)})^2}{2} \Big[\sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\, \frac{1}{\psi''(\overline{u}_\eta^*)} \Big]\, (u_P - u_P^*)^2 + \frac{1}{2} \sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\, (u_\eta^* - u_P^*)^2\, \psi''(\overline{u}_\eta^*), \end{aligned}$$

so this term may be absorbed in the first two ones under the CFL condition (2.49). $\square$

Chapter 3

The full Euler equations

3.1 Introduction

We address in this chapter the so-called Euler equations, which read:

$$\partial_t \rho + \operatorname{div}(\rho\, \boldsymbol{u}) = 0, \tag{3.1a}$$

$$\partial_t(\rho\, \boldsymbol{u}) + \operatorname{div}(\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \tag{3.1b}$$

$$\partial_t(\rho\, E) + \operatorname{div}(\rho\, E\, \boldsymbol{u}) + \operatorname{div}(p\, \boldsymbol{u}) = 0, \tag{3.1c}$$

$$p = (\gamma - 1)\, \rho\, e, \qquad E = \frac{1}{2}|\boldsymbol{u}|^2 + e, \tag{3.1d}$$

where t stands for the time, ρ, $\boldsymbol{u}$, p, E and e are the density, velocity, pressure, total energy and internal energy respectively, and $\gamma > 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0, T)$, where Ω is an open bounded connected subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, and $(0, T)$ is a finite time interval.

System (3.1) is complemented by initial conditions for ρ, e and $\boldsymbol{u}$, denoted by ρ_0, e_0 and $\boldsymbol{u}_0$ respectively, with $\rho_0 > 0$ and $e_0 > 0$, and by a boundary condition which we suppose to be:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$

at any time and *a.e.* on $\partial\Omega$, where $\boldsymbol{n}$ stands for the normal vector to the boundary.

Let us suppose that the solution is regular, and let E_k be the kinetic energy, defined by $E_k = \frac{1}{2}\,|\boldsymbol{u}|^2$. Taking the inner product of (3.1b) by $\boldsymbol{u}$ yields, after formal compositions of partial derivatives and using the mass balance (3.1a):

$$\partial_t(\rho E_k) + \mathrm{div}\big(\rho\, E_k\, \boldsymbol{u}\big) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0. \tag{3.2}$$

This relation is referred to as the kinetic energy balance. Substracting this relation from the total energy balance (3.1c), we obtain the internal energy balance equation:

$$\partial_t(\rho e) + \mathrm{div}(\rho e \boldsymbol{u}) + p\, \mathrm{div}(\boldsymbol{u}) = 0. \tag{3.3}$$

Since,

- thanks to the mass balance equation, the first two terms in the left-hand side of (3.3) may be recast as a transport operator: $\partial_t(\rho e) + \mathrm{div}(\rho e \boldsymbol{u}) = \rho\, [\partial_t e + \boldsymbol{u} \cdot \boldsymbol{\nabla} e]$,
- and, from the equation of state, the pressure vanishes when $e = 0$,

this equation implies, if $e \geq 0$ at $t = 0$ and with suitable boundary conditions, that e remains non-negative at all times.

The objective pursued in this work is to develop and study, from a theoretical point of view, an explicit scheme for the solution of (3.1). More precisely, we intend to build an explicit variant of pressure correction schemes that were developed and studied recently in the framework of the simulation of compressible flows at all speeds [17, 29, 26, 27], and implemented in the industrial open-source computer code ISIS [33]. Indeed, our initial motivation was to provide in the same software an efficient alternative of these schemes for quickly varying unstationary flows, with a characteristic Mach number in the range or greater than the unity. In order to remain stable in the incompressible limit, the starting-point algorithms are based on (*inf-sup* stable) staggered finite volume or finite element discretizations, and the present scheme thus also relies on these space approximations. In our approach, the upwinding techniques which are implemented for stability reasons are performed for each equation separately and with respect to the material velocity only. This is in contradiction with the most common strategy adopted for hyperbolic systems, where upwinding is built from the wave structure of the system (see *e.g.* [61, 19, 6] for surveys). However, it yields algorithms which are used in practice (see *e.g.* the so-called AUSM family of schemes [45, 44]), because of their generality (a closed-form solution of Riemann problems is not needed), their implementation simplicity and their efficiency, thanks to an easy construction of the fluxes at the cell faces.

Another salient feature of the propose scheme is that we discretize the internal energy balance (3.3) instead of the total energy balance (3.1c); this presents two advantages:

- first, it avoids the space discretization of the total energy, which is rather unnatural for staggered schemes since the degrees of freedom for the velocity and the scalar variables are not collocated,
- second, by a suitable discretization of (3.3), we obtain a scheme which ensures, “by construction”, the positivity of the internal energy.

However, for solutions with shocks, Equation (3.3) is not equivalent to (3.1c); more precisely speaking, at the locations of shocks, positive measures should appear, at the right-hand side of Equation (3.3). Discretizing (3.3) instead of (3.1c) may thus yield a scheme which does not compute the correct weak discontinuous solutions; in particular, the numerical solutions may present (smeared) shocks which do not satisfy the Rankine-Hugoniot conditions associated to (3.1c). The essential result of this chapter is to provide solutions to circumvent this problem. To this purpose, we closely mimic the above performed formal computation:

- we start from the discrete kinetic energy balance (3.3), and remark that the residual terms at the right-hand side do no tend to zero with the space and time steps (they are the discrete manifestations of the above mentioned measures),
- we thus compensate these residual terms by corrective terms in the internal energy balance.

We provide a theoretical justification of this process by showing that, in the 1D case, if the scheme is stable and converges to a limit (in a sense to be defined), this limit satisfies a weak form of (3.1c) which implies the correct Rankine-Hugoniot conditions.

This chapter is structured as follows. We begin with the presentation of the scheme (Section 3.2), then the discrete kinetic energy balance and the correction source terms of the internal energy equation are given in Section 3.3. The next section is dedicated to the proof, in 1D, of the consistency of the scheme (Section 3.4). We then present some numerical tests, to assess the behaviour of the algorithm (Section 3.5). Finally, the conclusion and perspectives are given in Section 3.6.

3.2 The scheme

We refer to Chapter 1, Section 1.2 for the space discretization. For the discretization in time, let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0,T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n = 0, \tag{3.4a}$$

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n e_\sigma^n + |K|\, p_K^n\, (\mathrm{div}\boldsymbol{u})_K^n = S_K^n, \tag{3.4b}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}, \tag{3.4c}$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$,

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n u_{\epsilon,i}^n + |D_\sigma|\, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0, \tag{3.4d}$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (3.4a) is obtained by the discretization of the mass balance equation (3.1a) over the primal mesh, and $F_{K,\sigma}^{n+1}$ stands for the mass flux across σ outward K, which, because of the impermeability condition, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad F_{K,\sigma}^n = |\sigma|\, \rho_\sigma^n\, u_{K,\sigma}^n, \tag{3.5}$$

where $u_{K,\sigma}^n$ is an approximation of the normal velocity to the face σ outward K. This latter quantity is defined by:

$$u_{K,\sigma}^n = \left| \begin{array}{l} u_{\sigma,i}^n\, \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\[2ex] \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{K,\sigma} \text{ in the CR and RT cases,} \end{array} \right. \tag{3.6}$$

where $\boldsymbol{e}^{(i)}$ denotes the i-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face

$\sigma = K|L$ is approximated by the upwind technique:

$$\rho_\sigma^n = \begin{vmatrix} \rho_K^n & \text{if } u_{K,\sigma}^n \geq 0, \\ \rho_L^n & \text{otherwise.} \end{vmatrix} \tag{3.7}$$

We now turn to the discrete momentum balance (3.4d), which is obtained by discretizing the momentum balance equation (3.1b) on the dual cells associated to the faces of the mesh. For the discretization of the time derivative, we must provide a definition for the values $\rho_{D_\sigma}^{n+1}$ and $\rho_{D_\sigma}^n$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\rm int},\ \text{for } k = n \text{ and } k = n+1, \qquad |D_\sigma|\ \rho_{D_\sigma}^k = |D_{K,\sigma}|\ \rho_K^k + |D_{L,\sigma}|\ \rho_L^k. \tag{3.8}$$

Let us then turn to the discretization of the convection term. The first task is to define the discrete mass flux through the dual face ϵ outward D_σ, denoted by $F_{\sigma,\epsilon}^n$; the guideline for its construction is that a finite volume discretization of the mass balance equation over the diamond cells, of the form

$$\forall \sigma \in \mathcal{E}, \qquad \frac{|D_\sigma|}{\delta t}\,(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n = 0, \tag{3.9}$$

must hold in order to be able to derive a discrete kinetic energy balance (see Section 3.3 below). For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [30]. In the case of the CR and RT schemes, for a dual face ϵ included in the primal cell K, this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the face and the cell) of the mass fluxes through the faces of K, *i.e.* the quantities $(F_{K,\sigma}^{n+1})_{\sigma \in \mathcal{E}(K)}$ appearing in the discrete mass balance (3.4a). We refer to [1, 17] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across that face is zero. Therefore, the values $u_{\epsilon,i}^{n+1}$ are only needed at the internal dual faces, and we make the upwind choice for their discretization:

$$\text{for } \epsilon = D_\sigma | D'_\sigma, \quad u^n_{\epsilon,i} = \left| \begin{array}{ll} u^n_{\sigma,i} & \text{if } F^n_{\sigma,\epsilon} \geq 0, \\ u^n_{\sigma',i} & \text{otherwise.} \end{array} \right. \tag{3.10}$$

The last term $(\boldsymbol{\nabla} p)^{n+1}_{\sigma,i}$ stands for the i-th component of the discrete pressure gradient at the face σ. The gradient operator is built as the transpose of the discrete operator for the divergence of the velocity, the discretization of which is based on the primal mesh. Let us denote the divergence of $\boldsymbol{u}^{n+1}$ over $K \in \mathcal{M}$ by $(\mathrm{div}\boldsymbol{u})^{n+1}_K$; its natural approximation reads:

$$\text{for } K \in \mathcal{M}, \qquad (\mathrm{div}\boldsymbol{u})^{n+1}_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, u^{n+1}_{K,\sigma}. \tag{3.11}$$

Consequently, the components of the pressure gradient are given by:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i} = \frac{|\sigma|}{|D_\sigma|} \, (p^{n+1}_L - p^{n+1}_K) \, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}, \tag{3.12}$$

this expression being derived thanks to the following duality relation with respect to the L^2 inner product:

$$\sum_{K \in \mathcal{M}} |K| \, p^{n+1}_K \, (\mathrm{div}\boldsymbol{u})^{n+1}_K + \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{\mathcal{S}}} |D_\sigma| \, u^{n+1}_{\sigma,i} \, (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i} = 0. \tag{3.13}$$

Note that, because of the impermeability boundary conditions, the discrete gradient is not defined at the external faces.

Equation (3.4b) is an approximation of the internal energy balance over the primal cell K. The positivity of the convection operator is ensured if we use an upwinding technique for this term [42]:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad e^n_\sigma = \left| \begin{array}{ll} e^n_K & \text{if } F^n_{K,\sigma} \geq 0, \\ e^n_L & \text{otherwise.} \end{array} \right.$$

The discrete divergence of the velocity, $(\mathrm{div}\boldsymbol{u})^n_K$, is defined by (3.11). The right-hand side, S^n_K, is derived using consistency arguments in the next section.

Finally, the initial approximations for ρ, e and $\boldsymbol{u}$ are given by the average of the initial conditions ρ_0, e_0 on the primal cells and $\boldsymbol{u}_0$ on the dual cells:

$$\begin{aligned}
&\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad \text{ and } e_K^0 = \frac{1}{|K|} \int_K e_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \\
&\text{for } 1 \leq i \leq d, \ \forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \quad u_{\sigma,i}^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} (\boldsymbol{u}_0(\boldsymbol{x}))_i \, \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.14}$$

The following positivity result is a classical consequence of the upwind choice in the mass balance equation.

Lemma 3.2.1 (Positivity of the density)**.** *Let ρ^0 be given by* (3.14)*. Then, since u_0 is assumed to be a positive function, $\rho^0 > 0$ and, under the* CFL *condition:*

$$\delta t \leq \frac{|K|}{\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \ \max(u_{K,\sigma}^n, 0)}, \qquad \forall K \in \mathcal{M}, \text{ for } 0 \leq n \leq N-1, \tag{3.15}$$

the solution to the scheme satisfies $\rho^n > 0$, for $1 \leq n \leq N$.

3.3 Discrete kinetic energy balance and corrective source terms

We begin by deriving a discrete kinetic energy balance equation. Equation (3.16) below is a discrete analogue of Equation (3.2), with an upwind discretization of the convection term. Its proof may be found in Chapter 2, Lemma 2.3.1.

Lemma 3.3.1 (Discrete kinetic energy balance)**.**
A solution to the system (3.4) *satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$ and $0 \leq n \leq N-1$:*

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \Big[\rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \Big] + \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n \, (u_{\epsilon,i}^n)^2 + |D_\sigma| \, (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} \, u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \tag{3.16}$$

with:

$$R^{n+1}_{\sigma,i} = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\rho^{n+1}_{D_\sigma}(u^{n+1}_{\sigma,i} - u^n_{\sigma,i})^2 + \frac{1}{2}\sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F^n_{D_\sigma,\epsilon})^-(u^n_{\sigma',i} - u^n_{\sigma,i})^2$$
$$- \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\mathcal{E}(D_\sigma)} (F^n_{D_\sigma,\epsilon})^-(u^n_{\sigma',i} - u^n_{\sigma,i})\,(u^{n+1}_{\sigma,i} - u^n_{\sigma,i}), \quad (3.17)$$

where, for $a \in \mathbb{R}$, $a^- \geq 0$ is defined by $a^- = -\min(a, 0)$.

The next step is now to define corrective terms in the internal energy balance, with the aim to recover a consistent discretization of the total energy balance. The first idea to do this could be just to sum the (discrete) kinetic energy balance with the internal energy balance: it is indeed possible for a collocated discretization. But here, we face the fact that the kinetic energy balance is associated to the dual mesh, while the internal energy balance is discretized on the primal one. The way to circumvent this difficulty is to remark that we do not really need a discrete total energy balance; in fact, we only need to recover (a weak form of) this equation when the mesh and time steps tend to zero. To this purpose, we choose the quantities (S^n_K) in such a way as to somewhat compensate the terms $(R^{n+1}_{\sigma,i})$ given by (3.17):

$$\forall K \in \mathcal{M},\ S^{n+1}_K = \sum_{i=1}^d S^{n+1}_{K,i} \text{ with} \qquad S^{n+1}_{K,i} = \frac{1}{2}\,\rho^{n+1}_K \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}^{(i)}_{\mathcal{S}}} \frac{|D_{K,\sigma}|}{\delta t}\left(u^{n+1}_{\sigma,i} - u^n_{\sigma,i}\right)^2$$
$$+ \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}^{(i)}_{\mathcal{S}},\ \epsilon\cap\bar{K}\neq\emptyset,\\ \epsilon=D_\sigma|D_{\sigma'},\, F^n_{\sigma,\epsilon}\leq 0}} \alpha_{K,\epsilon}\left[\frac{|F^n_{\sigma,\epsilon}|}{2}(u^n_{\sigma,i} - u^n_{\sigma',i})^2 + F^n_{\sigma,\epsilon}\left(u^{n+1}_{\sigma,i} - u^n_{\sigma,i}\right)\,\left(u^n_{\sigma',i} - u^n_{\sigma,i}\right)\right]. \quad (3.18)$$

The coefficient $\alpha_{K,\epsilon}$ is fixed to 1 if the face ϵ is included in K, and this is the only situation to consider for the RT and CR discretizations. For the MAC scheme, some dual faces are included in the primal cells, but some lie on their boundary; for such a boundary edge ϵ, we denote by $\mathcal{N}_\epsilon$ the set of cells M such that $\bar{M} \cap \epsilon \neq \emptyset$ (the cardinal of this set is always 4), and compute $\alpha_{K,\epsilon}$ by:

$$\alpha_{K,\epsilon} = \frac{|K|}{\sum_{M\in\mathcal{N}_\epsilon}|M|}. \quad (3.19)$$

For a uniform grid, this formula yields $\alpha_{K,\epsilon} = 1/4$.

The expression of the $(S_K^{n+1})_{K\in\mathcal{M}}$ is justified by the passage to the limit in the scheme (for a one-dimensional problem) performed in the next section. We note however here that:

$$\sum_{K\in\mathcal{M}} S_K^{n+1} - \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}_{\mathcal{S}}^{(i)}} R_{\sigma,i}^{n+1} = 0. \tag{3.20}$$

Indeed, the first part of $S_{K,i}^{n+1}$, thanks to the expression (3.8) of the density at the face $\rho_{D_\sigma}^{n+1}$, results from a dispatching of the first part of the residual over the two adjacent cells:

$$\frac{1}{2}\,\frac{|D_\sigma|}{\delta t}\,\rho_{D_\sigma}^{n+1}\,\left(u_{\sigma,i}^{n+1}-u_{\sigma,i}^{n}\right)^2 = \underbrace{\frac{1}{2}\,\frac{|D_{K,\sigma}|}{\delta t}\,\rho_K^{n+1}\,\left(u_{\sigma,i}^{n+1}-u_{\sigma,i}^{n}\right)^2}_{\text{affected to K}} + \underbrace{\frac{1}{2}\,\frac{|D_{L,\sigma}|}{\delta t}\,\rho_L^{n+1}\,\left(u_{\sigma,i}^{n+1}-u_{\sigma,i}^{n}\right)^2}_{\text{affected to L}}.$$

The same argument holds for the terms associated to the dual faces, which explains, in particular, the definition of the coefficients $\alpha_{K,\epsilon}$. The scheme thus conserves the integral of the total energy over the computational domain.

The definition (3.18) of $(S_K^{n+1})_{K\in\mathcal{M}}$ allows to prove that, under a CFL condition, the scheme preserves the positivity of e.

Lemma 3.3.2. *Let us suppose that, for $0\leq n\leq N-1$ and for all $K\in\mathcal{M}$, we have:*

$$\delta t \leq \frac{|K|}{\gamma \displaystyle\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,(u_{K,\sigma}^n)^+} \quad \textit{and} \quad \delta t \leq \frac{|D_{K,\sigma}|\,\rho_K^{n+1}}{\displaystyle\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\ \epsilon\cap\bar{K}\neq\emptyset} \alpha_{K,\epsilon}\,(F_{\sigma,\epsilon}^n)^-}, \qquad \forall\sigma\in\mathcal{E}(K). \tag{3.21}$$

Then the internal energy $(e^n)_{0\leq n\leq N}$ given by the scheme (3.4) *is positive.*

Proof. Let n such that $0\leq n\leq N$ be given, and let us assume that $e_K^n\geq 0$ for all $K\in\mathcal{M}$. Since, by assumption, the CFL condition (3.15) is satisfied, we have, by Lemma 3.2.1, $\rho_K^n>0$ and $\rho_K^{n+1}>0$, for all $K\in\mathcal{M}$. In the internal energy equation (3.4b), let us express the pressure thanks to the equation of state (3.4c) to obtain:

$$\frac{|K|}{\delta t}\rho_K^{n+1}e_K^{n+1} = \Big[\frac{|K|}{\delta t}\rho_K^n - \sum_{\sigma\in\mathcal{E}(K)} (F_{K,\sigma}^n)^+ - (\gamma-1)\,\rho_K^n \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\;(u_{K,\sigma}^n)^+\Big]\; e_K^n \\ + \sum_{\sigma\in\mathcal{E}(K)} (F_{K,\sigma}^n)^- e_L^n + (\gamma-1)\,\rho_K^n\, e_K^n \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\;(u_{K,\sigma}^n)^- + S_K^{n+1}. \quad (3.22)$$

Using the fact that, when $u_{K,\sigma}^n \geq 0$, the upwind density at the face is ρ_K^n, we have:

$$(F_{K,\sigma}^n)^+ + (\gamma-1)\,|\sigma|\;\rho_K^n\,(u_{K,\sigma}^n)^+ = \gamma|\sigma|\;\rho_K^n\,(u_{K,\sigma}^n)^+,$$

and hence Relation (3.22) reads:

$$\frac{|K|}{\delta t}\rho_K^{n+1}e_K^{n+1} = \Big[\frac{|K|}{\delta t} - \gamma \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\;(u_{K,\sigma}^n)^+\Big]\;\rho_K^n\, e_K^n \\ + \sum_{\sigma\in\mathcal{E}(K)} (F_{K,\sigma}^n)^- e_L^n + (\gamma-1)\,\rho_K^n\, e_K^n \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\;(u_{K,\sigma}^n)^- + S_K^{n+1}.$$

Let us suppose for a while that $S_K^{n+1} \geq 0$. Then we get $e_K^{n+1} > 0$ under the following CFL condition:

$$\delta t \leq \frac{|K|}{\gamma\;\sum_{\sigma\in\mathcal{E}(K)} |\sigma|(u_{K,\sigma}^n)^+}.$$

Let us now derive a condition for the non-negativity of the source term. Applying Young inequality to the last term of $S_{K,i}^{n+1}$, denoted by $(S_{K,i}^{n+1})_3$, we obtain

$$(S_{K,i}^{n+1})_3 \geq -\Big[\sum_{\substack{\epsilon\in\tilde{\mathcal{E}}_{\mathcal{S}}^{(i)},\ \epsilon\cap\bar{K}\neq\emptyset,\\ \epsilon=D_\sigma|D_{\sigma'},\, F_{\sigma,\epsilon}^n\leq 0}} \alpha_{K,\epsilon}\,\frac{|F_{\sigma,\epsilon}^n|}{2}\Big]\;\big(u_{\sigma,i}^{n+1} - u_{\sigma,i}^n\big)^2 \\ - \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}_{\mathcal{S}}^{(i)},\ \epsilon\cap\bar{K}\neq\emptyset,\\ \epsilon=D_\sigma|D_{\sigma'},\, F_{\sigma,\epsilon}^n\leq 0}} \alpha_{K,\epsilon}\,\frac{|F_{\sigma,\epsilon}^n|}{2}\,(u_{\sigma',i}^n - u_{\sigma,i}^n)^2.$$

Gathering all terms of $S_{K,i}^{n+1}$ yields:

$$S_{K,i}^{n+1} \geq \sum_{\sigma\in\mathcal{E}(K)} \frac{1}{2}\big(u_{\sigma,i}^{n+1} - u_{\sigma,i}^n\big)^2\Big[\frac{|D_{K,\sigma}|}{\delta t}\rho_K^{n+1} - \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\ \epsilon\cap\bar{K}\neq\emptyset} \alpha_{K,\epsilon}\,(F_{\sigma,\epsilon}^n)^-\Big],$$

thus $S_{K,i}^{n+1}$ is non-negative under the CFL condition:

$$\delta t \leq \frac{|D_{K,\sigma}|\, \rho_K^{n+1}}{\displaystyle\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\ \epsilon\cap\bar{K}\neq\emptyset} \alpha_{K,\epsilon}\,(F_{\sigma,\epsilon}^n)^-}, \qquad \forall \sigma \in \mathcal{E}(K),$$

which concludes the proof. □

3.4 Passing to the limit in the scheme

The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

The 1D version of the scheme which is studied in this section may be obtained from Scheme (3.4) by taking the MAC variant of the scheme, using only one horizontal stripe of grid cells, supposing that the vertical component of the velocity (the degrees of freedom of which are located on the top and bottom boundaries) vanishes, and that the measure of the vertical faces is equal to 1. For the sake of readability, however, we completely rewrite this 1D scheme, and, to this purpose, we first introduce some adaptations of the notations to the one dimensional case. For any $K \in \mathcal{M}$, we denote by h_K its length (so $h_K = |K|$); when we write $K = [\sigma\sigma']$, this means that either $K = (x_\sigma, x_{\sigma'})$ or $K = (x_{\sigma'}, x_\sigma)$; if we need to specify the order, *i.e.* $K = (x_\sigma, x_{\sigma'})$ with $x_\sigma < x_{\sigma'}$, then we write $K = [\overrightarrow{\sigma\sigma'}]$. For an interface $\sigma = K|L$ between two cells K and L, we define $h_\sigma = (h_K + h_L)/2$, so, by definition of the dual mesh, $h_\sigma = |D_\sigma|$. If we need to specify the order of the cells K and L, say K is left of L, then we write $\sigma = \overrightarrow{K|L}$. With these notations, the explicit scheme (3.4) may be written as follows in the one dimensional setting:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|}\int_K \rho_0(x)\,\mathrm{d}x, \qquad e_K^0 = \frac{1}{|K|}\int_K e_0(x)\,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|D_\sigma|}\int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \tag{3.23a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$

$$\frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{3.23b}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$
$$\frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^n e_{\sigma'}^n - F_\sigma^n e_\sigma^n + p_K^n (u_{\sigma'}^n - u_\sigma^n) = S_K^n, \tag{3.23c}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}, \tag{3.23d}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int},$$
$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^{n+1} - p_K^{n+1} = 0. \tag{3.23e}$$

The mass flux in the discrete mass balance equation is given, for $\sigma \in \mathcal{E}_{\rm int}$, by $F_\sigma^n = \rho_\sigma^n u_\sigma^n$, where the upwind approximation for the density at the face, ρ_σ^n, is defined by (3.7).

In the convection terms of the internal energy balance, the approximation for e_σ^n is upwind with respect to F_σ^n (*i.e.*, for $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}$, $e_\sigma^n = e_K^n$ if $F_\sigma^n \geq 0$ and $e_\sigma^n = e_L^n$ otherwise). The corrective term S_K^n reads, for $1 \leq n \leq N$ and $\forall K = [\sigma' \to \sigma]$:

$$S_K^n = \frac{|K|}{4\,\delta t}\, \rho_K^n \left[(u_\sigma^n - u_\sigma^{n-1})^2 + (u_{\sigma'}^n - u_{\sigma'}^{n-1})^2\right] + \frac{|F_K^{n-1}|}{2}\, (u_\sigma^{n-1} - u_{\sigma'}^{n-1})^2$$
$$- |F_K^{n-1}| (u_\sigma^n - u_\sigma^{n-1})\, (u_\sigma^{n-1} - u_{\sigma'}^{n-1}), \tag{3.24}$$

where the notation $K = [\sigma' \to \sigma]$ means that the flow goes from σ' to σ (*i.e.*, if $F_K^n \geq 0$, $K = [\overrightarrow{\sigma'\sigma}]$ and, if $F_K^n \leq 0$, $K = [\overrightarrow{\sigma\sigma'}]$). At the first time step, we thus set $S_K^0 = 0,\ \forall K \in \mathcal{M}$.

In the momentum balance equation, the application of the procedure described in Section 3.2 yields for the density associated to the dual cell D_σ with $\sigma = K|L$ and for the mass fluxes at the dual face located at the center of the mesh $K = [\overrightarrow{\sigma\sigma'}]$:

$$\text{for } k = n \text{ and } k = n+1,\ \rho_{D_\sigma}^k = \frac{1}{2\,|D_\sigma|}\, (|K|\, \rho_K^k + |L|\, \rho_L^k), \qquad F_K^n = \frac{1}{2}\, (F_\sigma^n + F_{\sigma'}^n), \tag{3.25}$$

and the approximation of the velocity at this face is upwind: $u_K^n = u_\sigma^n$ if $F_K^n \geq 0$ and $u_K^n = u_{\sigma'}^n$ otherwise.

Let a sequence of discretizations $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m \in \mathbb{N}}$ be given. We define the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ by $h^{(m)} = \sup_{K \in \mathcal{M}^{(m)}} h_K$. Let $\rho^{(m)}$, $p^{(m)}$, $e^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (3.23) with the mesh $\mathcal{M}^{(m)}$ and the time step $\delta t^{(m)}$. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho^{(m)}$, the pressure $p^{(m)}$, the internal energy $e^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0,T)$ by:

$$\rho^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (\rho^{(m)})_K^n \, \mathcal{X}_K(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{3.26a}$$

$$u^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} (u^{(m)})_\sigma^n \, \mathcal{X}_{D_\sigma}(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{3.26b}$$

$$p^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (p^{(m)})_K^n \, \mathcal{X}_K(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{3.26c}$$

$$e^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (e^{(m)})_K^n \, \mathcal{X}_K(x) \, \mathcal{X}_{(n,n+1]}(t), \tag{3.26d}$$

where $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{(n,n+1]}$ stand for the characteristic function of the intervals K, D_σ and $(t^n, t^{n+1}]$ respectively.

For discrete functions q and v defined on the primal and dual mesh, respectively, we define a discrete $\mathrm{L}^1((0,T);\mathrm{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\mathrm{int}}} |q_L^n - q_K^n|, \qquad \|v\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_{\mathrm{int}}} |v_{\sigma'}^n - v_\sigma^n|,$$

and a discrete $\mathrm{L}^1(\Omega;\mathrm{BV}((0,T)))$ norm by:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K \in \mathcal{M}} |K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma \in \mathcal{E}} |D_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

For the consistency result that we are seeking (Theorem 3.4.2 below), we have to assume that a sequence of discrete solutions $\left(\rho^{(m)}, p^{(m)}, u^{(m)}\right)_{m \in \mathbb{N}}$ satisfies $\rho^{(m)} > 0$, $p^{(m)} > 0$ and $e^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (3.15) is satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)^3$, *i.e.*:

$$\forall K \in \mathcal{M}^{(m)},\ \text{for } 0 \le n \le N^{(m)},\ \forall m \in \mathbb{N},$$
$$0 < (\rho^{(m)})_K^n \le C, \qquad 0 < (p^{(m)})_K^n \le C, \qquad 0 < (e^{(m)})_K^n \le C, \tag{3.27}$$

and

$$|(u^{(m)})_\sigma^n| \le C, \quad \forall \sigma \in \mathcal{E}^{(m)},\ \text{for } 0 \le n \le N^{(m)},\ \forall m \in \mathbb{N}, \tag{3.28}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0, e_0 and u_0 belong to $\mathrm{L}^\infty(\Omega)$. We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|p^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|e^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{3.29}$$

and:

$$\|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{3.30}$$

We are not able to prove the estimates (3.27)–(3.30) for the solutions of the scheme; however, such inequalities are satisfied by the “interpolates” (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* with a limited sequence of meshes and time steps).

A weak solution to the continuous problem satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\,\partial_t\varphi + \rho\,u\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{3.31a}$$

$$-\int_0^T \int_\Omega \Big[\rho\,u\,\partial_t\varphi + (\rho\,u^2 + p)\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,u_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{3.31b}$$

$$-\int_{\Omega\times(0,T)} \Big[\rho\,E\,\partial_t\varphi + (\rho\,E + p)\,u\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,E_0(x)\,\varphi(x,0)\,\mathrm{d}x = 0, \tag{3.31c}$$

$$p = (\gamma-1)\rho\,e, \quad E = \frac{1}{2}u^2 + e, \quad E_0 = \frac{1}{2}u_0^2 + e_0. \tag{3.31d}$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks* (*i.e.* shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions). This is the result we are seeking and which we state in Theorem 3.4.2. In order to prove this theorem, we need some definitions of interpolates of regular test functions on the primal and dual mesh.

Definition 3.4.1 (Interpolates on one-dimensional meshes)**.** *Let Ω be an open bounded interval of $\mathbb{R}$, let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$, and let $\mathcal{M}$ be a mesh over Ω. The interpolate $\varphi_{\mathcal{M}}$ of φ on the primal mesh $\mathcal{M}$ is defined by:*

$$\varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \varphi_K^{n+1}\, \mathcal{X}_K\, \mathcal{X}_{[t^n,t^{n+1})},$$

where, for $0 \leq n \leq N$ and $K \in \mathcal{M}$, we set $\varphi_K^n = \varphi(x_K, t^n)$, with x_K the mass center of K. The time discrete derivative of the discrete function $\varphi_{\mathcal{M}}$ is defined by:

$$\eth_t \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t}\, \mathcal{X}_K\, \mathcal{X}_{[t^n,t^{n+1})},$$

and its space discrete derivative by:

$$\eth_x \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\rm int}} \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}\, \mathcal{X}_{D_\sigma}\, \mathcal{X}_{[t^n,t^{n+1})}.$$

Let $\varphi_{\mathcal{E}}$ be an interpolate of φ on the dual mesh, defined by:

$$\varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \varphi_\sigma^{n+1}\, \mathcal{X}_{D_\sigma}\, \mathcal{X}_{[t^n,t^{n+1})},$$

where, for $1 \leq n \leq N$ and $\sigma \in \mathcal{E}$, we set $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$, with x_σ the abscissa of the interface σ. We also define the time and space discrete derivatives of this discrete *function by:*

$$\eth_t \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t} \, \mathcal{X}_{D_\sigma} \, \mathcal{X}_{[t^n, t^{n+1})},$$

$$\eth_x \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}}{h_K} \, \mathcal{X}_K \, \mathcal{X}_{[t^n, t^{n+1})}.$$

Finally, we define $\eth_x \varphi_{\mathcal{M},\mathcal{E}}$ *by:*

$$\begin{aligned} \eth_x \varphi_{\mathcal{M},\mathcal{E}} = \sum_{n=0}^{N-1} \quad \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_\sigma^{n+1}}{h_K/2} \, \mathcal{X}_{D_{K,\sigma}} \, \mathcal{X}_{[t^n, t^{n+1})} \\ + \frac{\varphi_{\sigma'}^{n+1} - \varphi_K^{n+1}}{h_K/2} \, \mathcal{X}_{D_{K,\sigma'}} \, \mathcal{X}_{[t^n, t^{n+1})}. \end{aligned}$$

We are now in position to state the following result.

Theorem 3.4.2 (Consistency of the one-dimensional explicit scheme)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$, $p_0 \in \mathrm{BV}(\Omega)$, $e_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and let $(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)})_{m \in \mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (3.27)–(3.30) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^4$, for $1 \le p < \infty$, to $(\bar\rho, \bar p, \bar e, \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^4$.*

Then the limit $(\bar\rho, \bar p, \bar e, \bar u)$ satisfies the system (3.31)*.*

Proof. It is clear that with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The fact that the limit satisfies the weak mass balance equation (3.31a) and the weak momentum balance equation (3.31b) is proven in Chapter 2, Theorem 2.4.2. The proof of this theorem is thus obtained by passing to the limit in the scheme, in the internal and the kinetic energy balance equations.

Let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$. Let $m \in \mathbb{N}$, $\mathcal{M}^{(m)}$ and $\delta t^{(m)}$ be given. Dropping for short the superscript ${}^{(m)}$, let $\varphi_{\mathcal{M}}$ be the interpolate of φ on the primal mesh and let $\eth_t \varphi_{\mathcal{M}}$ and $\eth_x \varphi_{\mathcal{M}}$ be its time and space discrete derivatives in the sense of Definition 3.4.1. Thanks to the regularity of φ, these functions respectively converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \geq 1$ (including $r = +\infty$), to φ, $\partial_t \varphi$ and $\partial_x \varphi$ respectively. In addition, $\varphi_{\mathcal{M}}(\cdot, 0)$ (which, for $K \in \mathcal{M}$ and $x \in K$, is equal to $\varphi_K^1 = \varphi(x_K, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \geq 1$.

We also define $\varphi_{\mathcal{E}}$, $\eth_t \varphi_{\mathcal{E}}$ and $\eth_x \varphi_{\mathcal{E}}$, as, respectively, the interpolate of φ on the dual mesh and its discrete time and space derivatives, still in the sense of Definition 3.4.1; once again thanks to the regularity of φ, these functions converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \geq 1$, to φ, $\partial_t \varphi$ and $\partial_x \varphi$ respectively. As for the interpolate on the primal mesh, $\varphi_{\mathcal{E}}(\cdot, 0)$ (which, for $\sigma \in \mathcal{E}$ and $x \in D_\sigma$, is equal to $\varphi_\sigma^1 = \varphi(x_\sigma, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \geq 1$.

Since the support of φ is compact in $\Omega \times [0, T)$, for m large enough, the interpolates of φ vanish on the boundary cells and at the last time step(s); hereafter, we systematically assume that we are in this case.

On one hand, let us multiply Equation (3.4b) by $\delta t\, \varphi_K^{n+1}$, and sum the result for $0 \leq n \leq N-1$ and $K \in \mathcal{M}$. On the second hand, let us multiply the discrete kinetic energy balance (3.16) by $\delta t\, \varphi_\sigma^{n+1}$, and sum the result over for $0 \leq n \leq N-1$ and $\sigma \in \mathcal{E}_{\mathrm{int}}$. Finally, adding the two obtained relations, we get:

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + \tilde{T}_1^{(m)} + \tilde{T}_2^{(m)} + \tilde{T}_3^{(m)} = S^{(m)} - \tilde{R}^{(m)} \tag{3.32}$$

where:

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t} \left[\rho_K^{n+1}\, e_K^{n+1} - \rho_K^n\, e_K^n\right] \varphi_K^{n+1},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[\rho_{\sigma'}^n\, e_{\sigma'}^n\, u_{\sigma'}^n - \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\right] \varphi_K^{n+1},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} p_K^n\, (u_{\sigma'}^n - u_\sigma^n)\, \varphi_K^{n+1},$$

$$\tilde{T}_1^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_\sigma^n (u_\sigma^n)^2\right] \varphi_\sigma^{n+1},$$

$$\tilde{T}_2^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}} \left[F_L^n\, (u_L^n)^2 - F_K^n\, (u_K^n)^2\right] \varphi_\sigma^{n+1},$$

$$\tilde{T}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\rm int}} (p_L^{n+1} - p_K^{n+1})\, u_\sigma^{n+1}\, \varphi_\sigma^{n+1},$$

$$S^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^n\, \varphi_K^{n+1}, \qquad \tilde{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\rm int}} R_\sigma^{n+1}\, \varphi_\sigma^{n+1},$$

and the quantities S_K^{n+1} and R_σ^{n+1} are given by Equation (3.24) and (the one-dimensional version of) Equation (3.17) respectively.

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |K|\, \rho_K^n\, e_K^n\, \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{K \in \mathcal{M}} |K|\, \rho_K^0\, e_K^0\, \varphi_K^1,$$

so that:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, e^{(m)} \eth_t \varphi_{\mathcal{M}}\, \mathrm{d}x\, \mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\ (e^{(m)})^0(x)\ \varphi_{\mathcal{M}}(x,0)\, \mathrm{d}x.$$

The boundedness of ρ_0, e_0 and the definition (3.23a) of the initial conditions for the scheme ensures that the sequences $((\rho^{(m)})^0)_{m \in \mathbb{N}}$ and $((e^{(m)})^0)_{m \in \mathbb{N}}$ converge to ρ_0 and e_0 respectively in $\mathrm{L}^r(\Omega)$ for $r \geq 1$. Since, by assumption, the sequence of discrete solutions and of the interpolate time derivatives converge in $\mathrm{L}^r\big(\Omega \times [0,T)\big)$ for $r \geq 1$, we thus obtain:

$$\lim_{m \to +\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar{\rho}\, \bar{e}\, \partial_t \varphi\, \mathrm{d}x\, \mathrm{d}t - \int_\Omega \rho_0(x)\ e_0(x)\ \varphi(x,0)\, \mathrm{d}x.$$

Reordering the sums in $T_2^{(m)}$, we get:

$$T_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}} \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\, (\varphi_L^{n+1} - \varphi_K^{n+1}).$$

Using the fact that $h_\sigma = |D_\sigma|$, this relation reads:

$$T_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}} |D_\sigma|\, \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

thus $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with:

$$\mathcal{T}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \Big[|D_{K,\sigma}|\, \rho_K^n\, e_K^n + |D_{L,\sigma}|\, \rho_L^n\, e_L^n\Big]\, u_\sigma^n\, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

$$\mathcal{R}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \Big[|D_\sigma|\, \rho_\sigma^n\, e_\sigma^n - |D_{K,\sigma}|\, \rho_K^n\, e_K^n - |D_{L,\sigma}|\, \rho_L^n\, e_L^n\Big]\, u_\sigma^n\, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}.$$

The first expression reads:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, e^{(m)}\, u^{(m)}\, \eth_x \varphi_{\mathcal{M}} \,\mathrm{d}x \,\mathrm{d}t,$$

and thus, thanks to the convergence assumptions for the solution:

$$\lim_{m\to+\infty} \mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \bar e\, \bar u\, \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Let us make a change of notation for the orientation of σ in such a way that $\rho_\sigma^n = \rho_K^n$ and $e_\sigma^n = e_K^n$ (in other words, we choose to call K the downwind cell to σ instead of the left cell, which we denote by $\sigma = K \to L$). We thus get, with $C_\varphi = \|\partial_x \varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}$:

$$|\mathcal{R}_2^{(m)}| \leq C_\varphi \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K\to L\in\mathcal{E}} |D_{L,\sigma}|\, \Big|\rho_K^n\, e_K^n - \rho_L^n\, e_L^n\Big|\, |u_\sigma^n|.$$

Applying the identity $2\,(ab - cd) = (a-c)(b+d) + (a+c)(b-d)$, which holds for any $\{a,b,c,d\} \subset \mathbb{R}$, to the quantity $\rho_K^n\, e_K^n - \rho_L^n\, e_L^n$, we obtain:

$$\begin{aligned} |\mathcal{R}_2^{(m)}| \leq C_\varphi\, h^{(m)}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \Big[\|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|e^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \\ + \|e^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\Big], \end{aligned}$$

and thus $|\mathcal{R}_2^{(m)}|$ tends to zero when m tends to $+\infty$.

For the term $\tilde{T}_1^{(m)}$, the definition (3.25) of ρ_{D_σ} yields:

$$\tilde{T}_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}} \Big[|D_{K,\sigma}|\,\rho_K^n + |D_{L,\sigma}|\,\rho_L^n\Big]\,u_\sigma^n\,\frac{\varphi_K^{n+1}-\varphi_K^n}{\delta t} - \sum_{\sigma=K|L\in\mathcal{E}} \Big[|D_{K,\sigma}|\,\rho_K^0 + |D_{L,\sigma}|\,\rho_L^0\Big]\,u_\sigma^0\,\varphi_K^1,$$

so, by similar arguments as for the term $T_1^{(m)}$, we get:

$$\lim_{m\to+\infty} \tilde{T}_1^{(m)} = -\int_0^T \int_\Omega \frac{1}{2}\,\bar\rho\,\bar u^2\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \frac{1}{2}\,\rho_0(x)\,u_0(x)^2\,\varphi(x,0)\,\mathrm{d}x.$$

Let us now to the term $\tilde{T}_2^{(m)}$. Reordering the sums, we get:

$$\tilde{T}_2^{(m)} = -\frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} F_K^n\,(u_K^n)^2\,(\varphi_{\sigma'}^{n+1}-\varphi_\sigma^{n+1}),$$

so, by the definition of the mass flux at the dual edges:

$$\tilde{T}_2^{(m)} = -\frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\,(u_K^n)^2\,(\varphi_{\sigma'}^{n+1}-\varphi_\sigma^{n+1}),$$

where we recall that u_K^n is equal to either u_σ^n or $u_{\sigma'}^n$, depending on the sign of F_K^n. Let us write $\tilde{T}_2^{(m)} = \tilde{\mathcal{T}}_2^{(m)} + \tilde{\mathcal{R}}_2^{(m)}$, with:

$$\tilde{\mathcal{T}}_2^{(m)} = -\frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \rho_K^n\,\big[(u_\sigma^n)^3 + (u_{\sigma'}^n)^3\big]\,(\varphi_{\sigma'}^{n+1}-\varphi_\sigma^{n+1}).$$

We have:

$$\tilde{\mathcal{T}}_2^{(m)} = -\int_0^T \int_\Omega \frac{1}{2}\,\rho^{(m)}\,(u^{(m)})^3\,\eth_x\varphi_{\mathcal{E}}\,\mathrm{d}x\,\mathrm{d}t,$$

and hence:

$$\lim_{m\to+\infty} \tilde{\mathcal{T}}_2^{(m)} = -\int_0^T \int_\Omega \frac{1}{2}\,\bar\rho\,\bar u^3\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

The remainder term reads:

$$\tilde{\mathcal{R}}_2^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \Big[(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\,(u_K^n)^2 - \rho_K^n \Big((u_\sigma^n)^3 + (u_{\sigma'}^n)^3 \Big) \Big]\,(\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

Possibly exchanging the notations for the faces of K, we may write $u_K^n = u_\sigma^n$, to obtain:

$$\tilde{\mathcal{R}}_2^{(m)} = -\frac{\varepsilon}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma']\in\mathcal{M}} \Big[(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\,(u_\sigma^n)^2 - \rho_K^n \Big((u_\sigma^n)^3 + (u_{\sigma'}^n)^3 \Big) \Big]\,(\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

with $\varepsilon = \pm 1$. Since, for $0 \leq n \leq N-1$ and $K \in \mathcal{M}$,

$$\begin{aligned} &(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\,(u_\sigma^n)^2 - \rho_K^n \big((u_\sigma^n)^3 + (u_{\sigma'}^n)^3 \big) = \\ &\qquad - (\rho_K^n - \rho_\sigma^n)\,(u_\sigma^n)^3 + \rho_K^n\, u_{\sigma'}^n\,(u_\sigma^n + u_{\sigma'}^n)\,(u_\sigma^n - u_{\sigma'}^n) - (\rho_K^n - \rho_{\sigma'}^n)\, u_{\sigma'}^n\,(u_\sigma^n)^2, \end{aligned}$$

we have:

$$\begin{aligned} |\tilde{\mathcal{R}}_2^{(m)}| \leq C_\varphi\, h^{(m)} \Big[&\|u^{(m)}\|^3_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|\rho\|_{\mathcal{T},x,\mathrm{BV}} \\ &+ \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \Big], \end{aligned}$$

where the real number C_φ only depends on φ. Hence $|\tilde{\mathcal{R}}_2^{(m)}|$ tends to zero when m tends to $+\infty$.

We now turn to $T_3^{(m)}$ and $\tilde{T}_3^{(m)}$. By a change in the notation of the time exponents, using the fact that φ_σ vanishes at the last time step(s), we get:

$$\tilde{T}_3^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} (p_L^n - p_K^n)\, u_\sigma^n\, \varphi_\sigma^n = \tilde{\mathcal{T}}_3^{(m)} + \tilde{\mathcal{R}}_3^{(m)},$$

with:

$$\tilde{\mathcal{T}}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, u_\sigma^n\, \varphi_\sigma^{n+1},$$

$$\tilde{\mathcal{R}}_3^{(m)} = \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^0 - p_K^0)\, u_\sigma^0\, \varphi_\sigma^1 + \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, u_\sigma^n\, (\varphi_\sigma^n - \varphi_\sigma^{n+1}).$$

We have, thanks to the regularity of φ:

$$|\tilde{\mathcal{R}}_3^{(m)}| \le C_\varphi\, \delta t^{(m)} \Big[\|(u^{(m)})^0\|_{\mathrm{L}^\infty(\Omega)}\, \|(p^{(m)})^0\|_{\mathrm{BV}(\Omega)} + \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|p^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \Big].$$

Therefore, invoking the regularity of the initial conditions, this term tends to zero when m tends to $+\infty$. We now have for the other terms, reordering the summations:

$$\begin{aligned} T_3^{(m)} + \tilde{\mathcal{T}}_3^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\, u_\sigma^n\, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}) + p_K^n\, u_{\sigma'}^n\, (\varphi_{\sigma'}^{n+1} - \varphi_K^{n+1}) \\ = -\int_0^T \int_\Omega p^{(m)}\; u^{(m)}\; \eth_x \varphi_{\mathcal{M},\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

So, since $\eth_x \varphi_{\mathcal{M},\mathcal{E}}$ converges to $\partial_x \varphi$ in $\mathrm{L}^r(\Omega \times (0,T))$ for any $r \ge 1$, we get:

$$\lim_{m\to+\infty} T_3^{(m)} + \tilde{\mathcal{T}}_3^{(m)} = -\int_0^T \int_\Omega \bar p\; \bar u\; \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Finally, it now remains to check that $\lim_{m\to+\infty} S^{(m)} - \tilde{R}^{(m)} = 0$. Let us write this quantity as $S^{(m)} - \tilde{R}^{(m)} = \mathcal{R}_1^{(m)} + \mathcal{R}_2^{(m)}$ where, using $S_K^0 = 0, \forall K \in \mathcal{M}$:

$$\mathcal{R}_1^{(m)} = \sum_{n=0}^{N-1} \delta t \big[\sum_{K\in\mathcal{M}} S_K^{n+1} \varphi_K^{n+1} - \sum_{\sigma\in\mathcal{E}} R_\sigma^{n+1} \varphi_\sigma^{n+1} \big],$$

$$\mathcal{R}_2^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{K\in\mathcal{M}} S_K^n\, (\varphi_K^{n+1} - \varphi_K^n).$$

First, we prove that $\lim_{m\to+\infty} \mathcal{R}_1^{(m)} = 0$. Gathering and reordering sums, we obtain

$\mathcal{R}_1^{(m)} = \mathcal{R}_{1,1}^{(m)} + \mathcal{R}_{1,2}^{(m)} + \mathcal{R}_{1,3}^{(m)}$ with

$$\begin{aligned}
\mathcal{R}_{1,1}^{(m)} &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{\sigma=K|L\in\mathcal{E}} \Big[\frac{|D_{K,\sigma}|}{\delta t}\rho_K^{n+1}(u_\sigma^{n+1}-u_\sigma^n)^2(\varphi_K^{n+1}-\varphi_\sigma^{n+1}) \\
&\qquad\qquad + \frac{|D_{L,\sigma}|}{\delta t}\rho_L^{n+1}(u_\sigma^{n+1}-u_\sigma^n)^2(\varphi_L^{n+1}-\varphi_\sigma^{n+1})\Big], \\
\mathcal{R}_{1,2}^{(m)} &= \frac{1}{2}\sum_{n=0}^{N-1}\delta t \sum_{K\in\mathcal{M}} |F_K^n|\,(u_\sigma^n - u_{\sigma'}^n)^2\,(\varphi_K^{n+1}-\varphi_\sigma^{n+1}), \\
\mathcal{R}_{1,3}^{(m)} &= \sum_{n=0}^{N-1}\delta t \sum_{K=[\sigma'\to\sigma]\in\mathcal{M}} F_K^n\,(u_{\sigma'}^n - u_\sigma^n)\,(u_\sigma^{n+1}-u_\sigma^n)\,(\varphi_K^{n+1}-\varphi_\sigma^{n+1}).
\end{aligned}$$

We thus obtain:

$$|\mathcal{R}_{1,1}^{(m)}| \leq h^{(m)}\, C_\varphi\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}},$$

and
$$|\mathcal{R}_{1,2}^{(m)}|+|\mathcal{R}_{1,3}^{(m)}| \leq h^{(m)}\, C_\varphi\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}},$$

so all these terms tend to zero. The fact that $|\mathcal{R}_2^{(m)}|$ behaves as $\delta t^{(m)}$ my be proven by very similar arguments.

Gathering the limits of all terms concludes the proof. □

3.5 Numerical results

We assess in this section the behaviour of the scheme on various test cases. To this purpose, we address the five Riemann problems studied in [61, Chapter 4]. More precisely, we perform a detailed study of the test referred in [61, Chapter 4] as Test 3, and give the results obtained on the other tests for the sake of completeness.

3.5.1 Test 3

In this test, the chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = 0 \\ p_L = 1000 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 1 \\ u_R = 0 \\ p_R = 0.001 \end{bmatrix}.$$

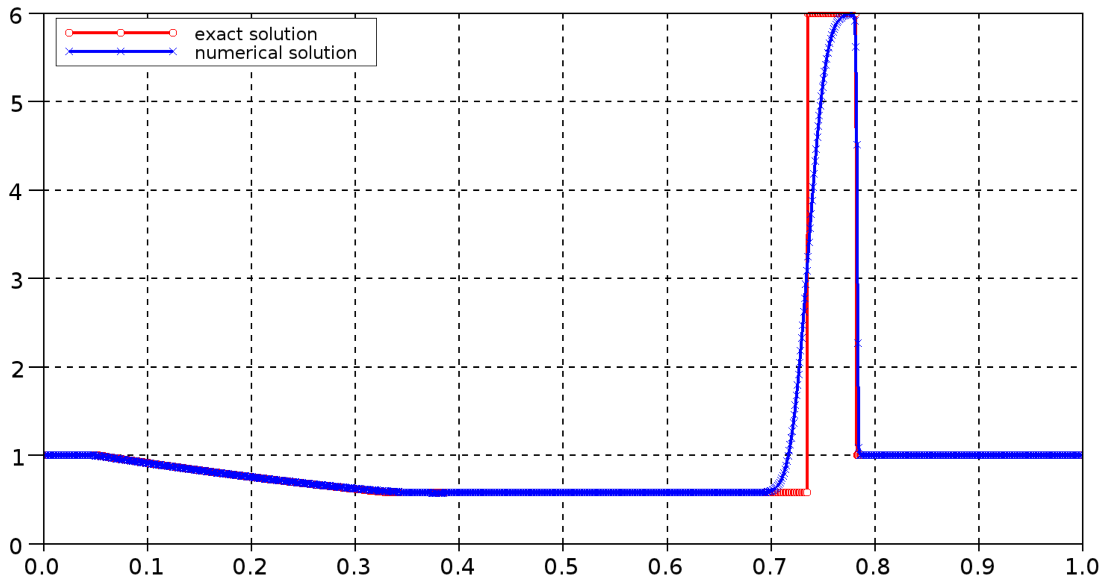


Figure 3.1: Test 3 – $h = 0.001$ and $\delta t = h/100$ – Density at $t = 0.012$.

The computational domain is $\Omega = (0, 1)$ and the final time is $T = 0.012$. The (known) analytical solution of this problem consists in a rarefaction wave, travelling to the left, and a shock wave, travelling to the right, separated by the constant discontinuity.

3.5.1.1 Results

The density, pressure, internal energy and velocity obtained at $t = 0.012 = T$ with $h = 0.001$ and $\delta t = h/100$ are shown on Figures 3.1, 3.2, 3.3 and 3.4 respectively. We observe that the scheme is rather diffusive especially for contact discontinuities for which the beneficial compressive effect of the shocks does not apply. More accurate variants may certainly be derived, using for instance MUSCL-like techniques; this work is underway.

In addition, we perform a convergence study, successively dividing by two the space and time steps (so keeping the CFL number constant). The difference between the computed and analytical solution at $t = 0.025$, measured in $\mathrm{L}^1(\Omega)$ norm, are reported in the following table.

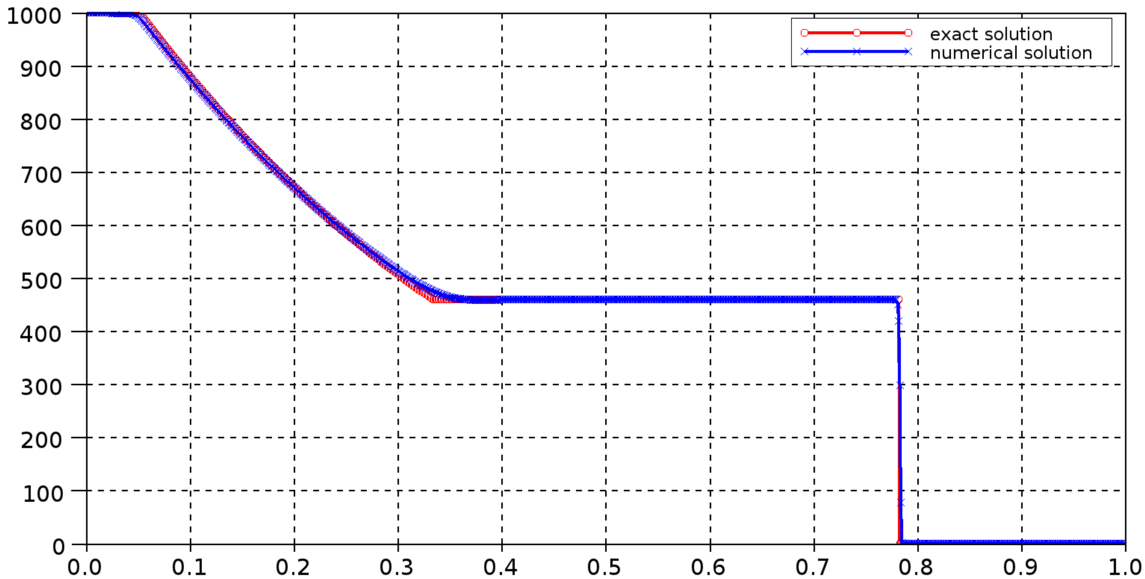


Figure 3.2: Test 3 – $h = 0.001$ and $\delta t = h/100$ – Pressure at $t = 0.012$.

space step	$h_0 = 0.001$	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
$\|\rho - \bar{\rho}\|_{\mathrm{L}^1(\Omega)}$	0.0651	0.0455	0.0310	0.0217	0.0153
$\|p - \bar{p}\|_{\mathrm{L}^1(\Omega)}$	1.87	1.05	0.530	0.284	0.164
$\|u - \bar{u}\|_{\mathrm{L}^1(\Omega)}$	0.0967	0.0536	0.0258	0.0134	0.00795

We measure a convergence rate which is slightly lower to 1 for the variables which are constant through the contact discontinuity (*i.e.* p and u), and equal to 1/2 for ρ.

Finally, we test the behaviour of the scheme obtained when setting to zero the corrective terms in the internal energy balance. Results with $h = 0.001$ and $\delta t = h/100$ are reported on Figures 3.5–3.8. From further numerical experiments with more and more refined meshes, it seems that the scheme converge, but to a limit which is not a weak solution to the Euler system: indeed, the Rankine-Hugoniot condition applied to the total energy balance, with the states obtained numerically, yields a right shock velocity slightly greater than the analytical solution one, while the same shock velocity obtained numerically is clearly lower.

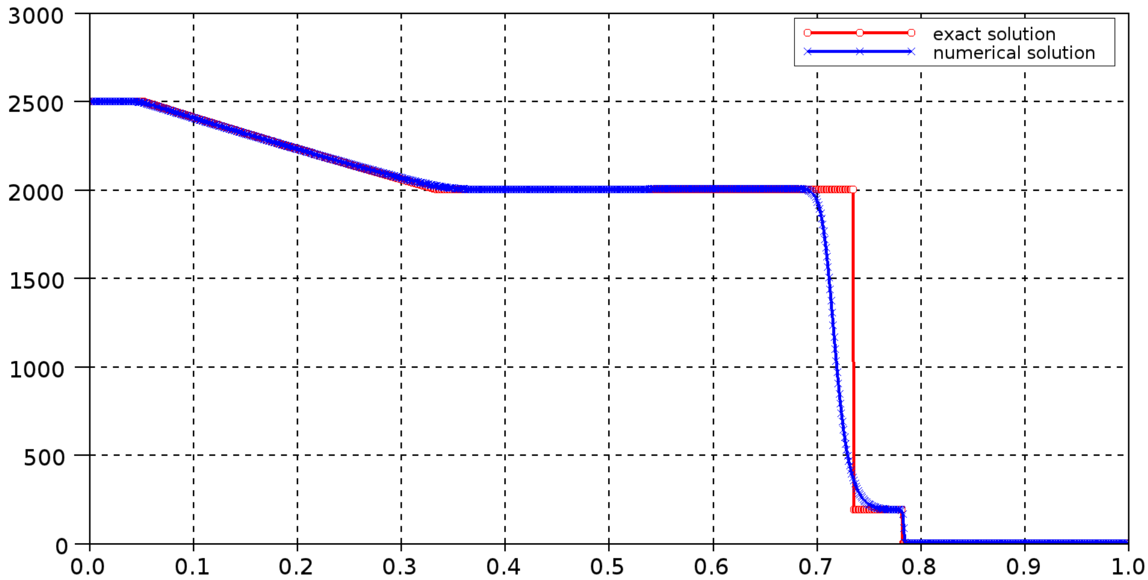


Figure 3.3: Test 3 – $h = 0.001$ and $\delta t = h/100$ – Internal energy at $t = 0.012$.

3.5.1.2 On a naive scheme

We also test the "naive" explicit scheme obtained by evaluating all the terms, except in time-derivative one, at time t^n. In the one dimensional setting and with the same notations as in Section 3.4, this scheme thus reads:

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{3.33a}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \quad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^n - p_K^n = 0, \tag{3.33b}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^n e_{\sigma'}^n - F_\sigma^n e_\sigma^n + p_K^n (u_{\sigma'}^n - u_\sigma^n) = S_K^{n+1}, \tag{3.33c}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}. \tag{3.33d}$$

Hereafter and on the figure captions, this scheme is referred to by the $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme (since the pressure is updated after the computation of the velocity rather than

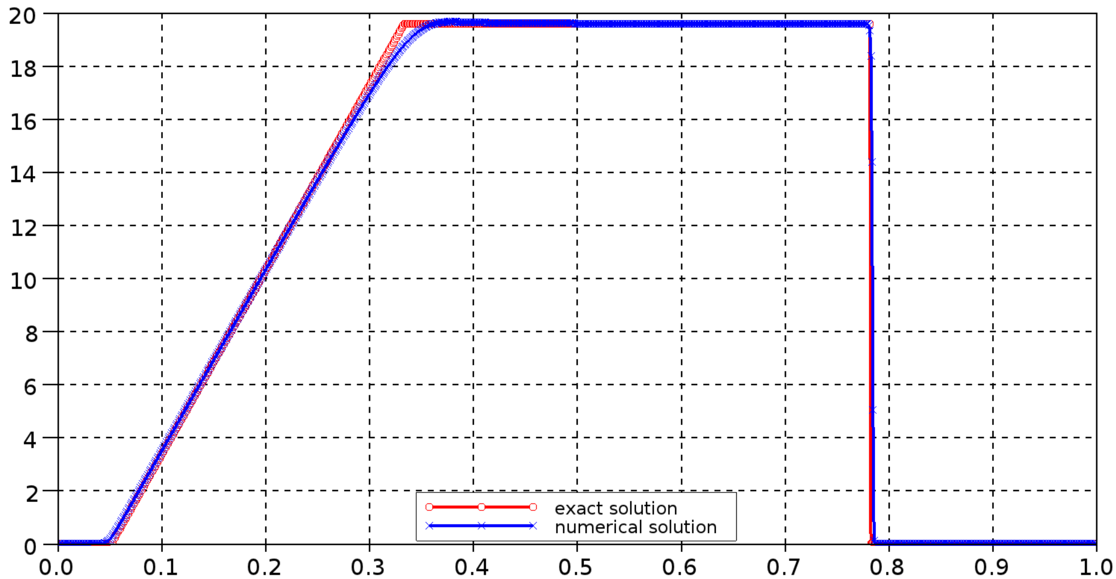


Figure 3.4: Test 3 – $h = 0.001$ and $\delta t = h/100$ – Velocity at $t = 0.012$.

after the computation of the density). Note that we are able, for this scheme also, to prove a consistency result similar to Theorem 3.4.2.

The computed density, pressure, internal energy and velocity at time $T = 0.012$ are plotted on figures 3.9, 3.10, 3.11 and 3.12 respectively. From these results, it appears clearly that the $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme generates discontinuities in the rarefaction wave, and further experiments show that this phenomenon is not cured by a reduction of the time and space step.

3.5.2 Test 1

In this test, the chosen left and right states are given by:

$$\text{left state:} \quad \begin{bmatrix} \rho_L = 1 \\ u_L = 0 \\ p_L = 1 \end{bmatrix}; \qquad \text{right state:} \quad \begin{bmatrix} \rho_R = 0.125 \\ u_R = 0 \\ p_R = 0.1 \end{bmatrix}.$$

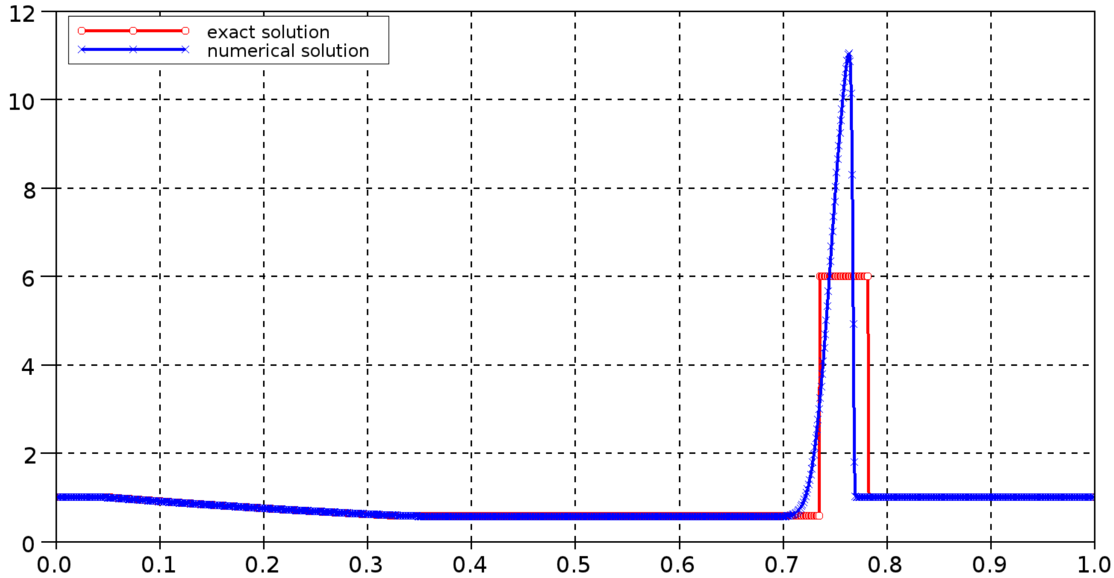


Figure 3.5: Test 3, without corrective terms – $h = 0.001$ and $\delta t = h/100$ – Density at $t = 0.012$.

The computational domain is $\Omega = (0, 1)$ and the final time is $T = 0.25$. The (known) analytical solution of this type of problem consists in two genuinely nonlinear waves (*i.e.* rarefaction or shock waves) separated by a contact discontinuity. For the initial data chosen in this section, the left wave is a rarefaction wave and the right one is a shock.

Results obtained with $h = 0.001$ and $\delta t = h/6$ at $t = T$ are shown on Figures 3.13–3.16.

3.5.3 Test 2

The chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = -2 \\ p_L = 0.4 \end{bmatrix} ; \qquad \text{right state: } \begin{bmatrix} \rho_R = 1 \\ u_R = 2 \\ p_R = 0.4 \end{bmatrix} .$$

The computational domain is $\Omega = (0, 1)$ and the final time is $T = 0.15$. Both left and right waves are rarefaction waves.

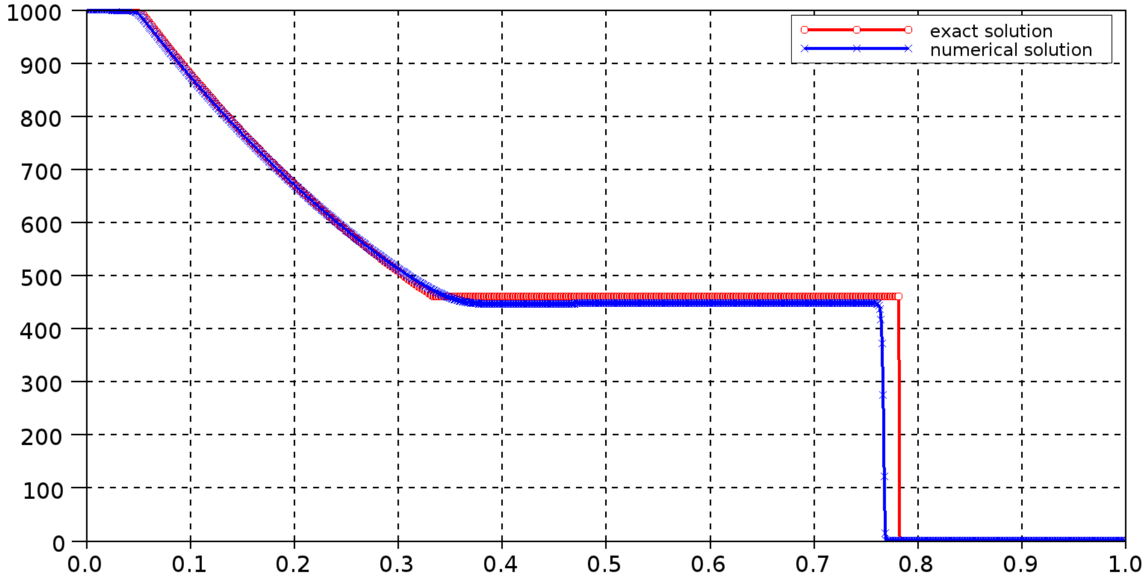


Figure 3.6: Test 3, without corrective terms – $h = 0.001$ and $\delta t = h/100$ – Pressure at $t = 0.012$.

Results obtained with $h = 0.001$ and $\delta t = h/5$ at $t = T$ are shown on Figures 3.17–3.20.

3.5.4 Test 4

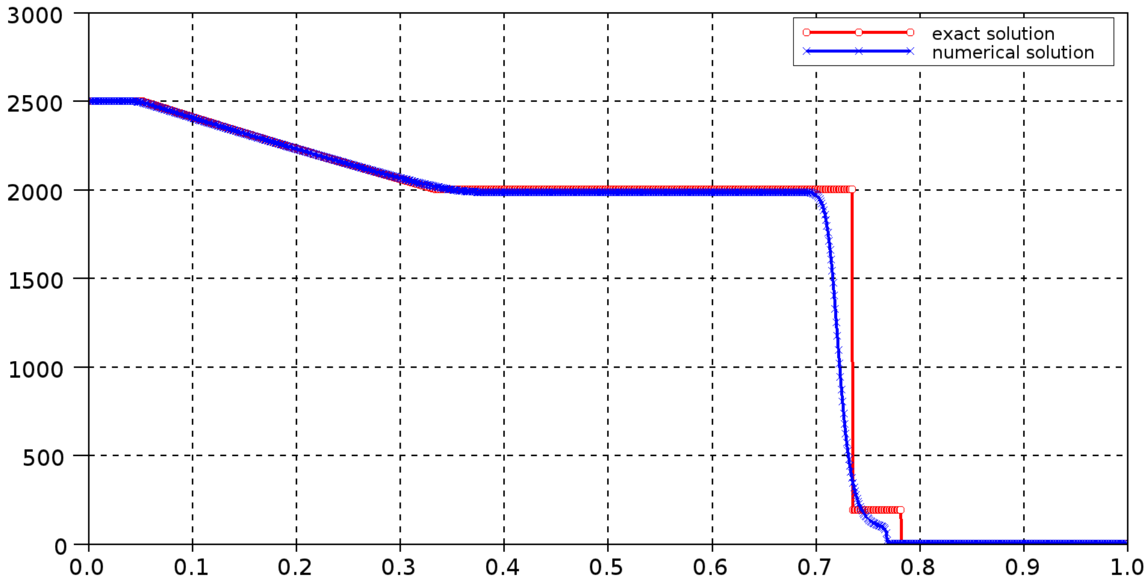


Figure 3.7: Test 3, without corrective terms – $h = 0.001$ and $\delta t = h/100$ – Internal energy at $t = 0.012$.

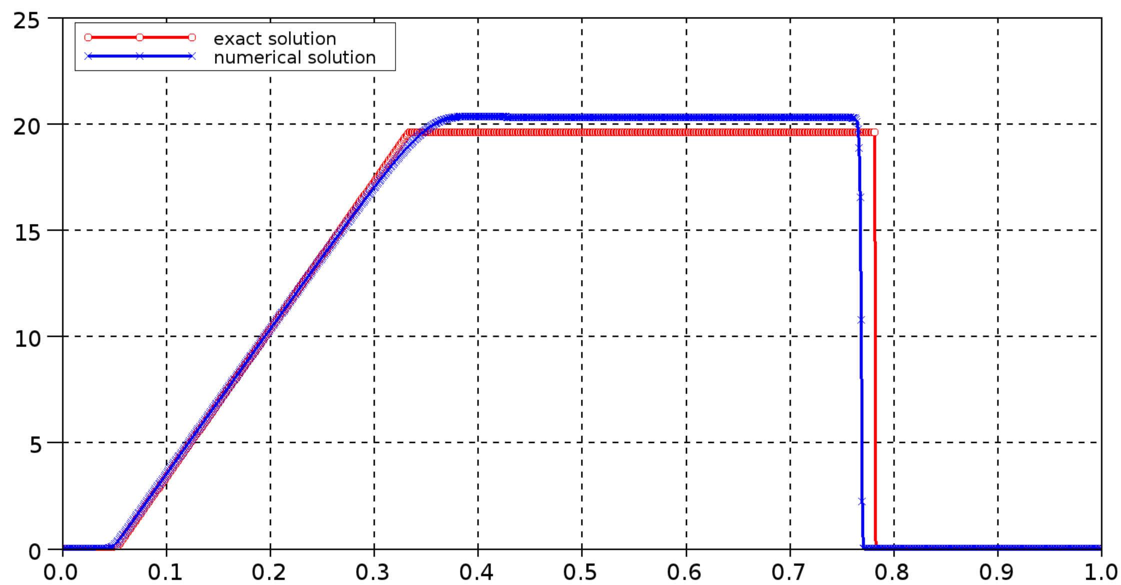


Figure 3.8: Test 3, without corrective terms – $h = 0.001$ and $\delta t = h/100$ – Velocity at $t = 0.012$.

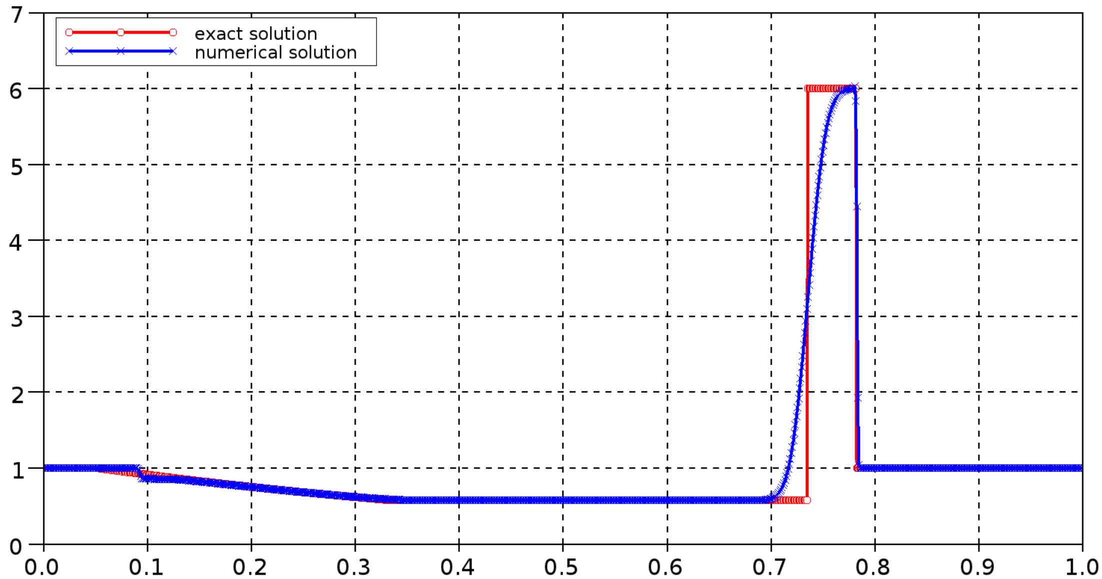


Figure 3.9: Test 3, $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme – $h = 0.001$ and $\delta t = h/100$ – Density at $t = 0.012$.

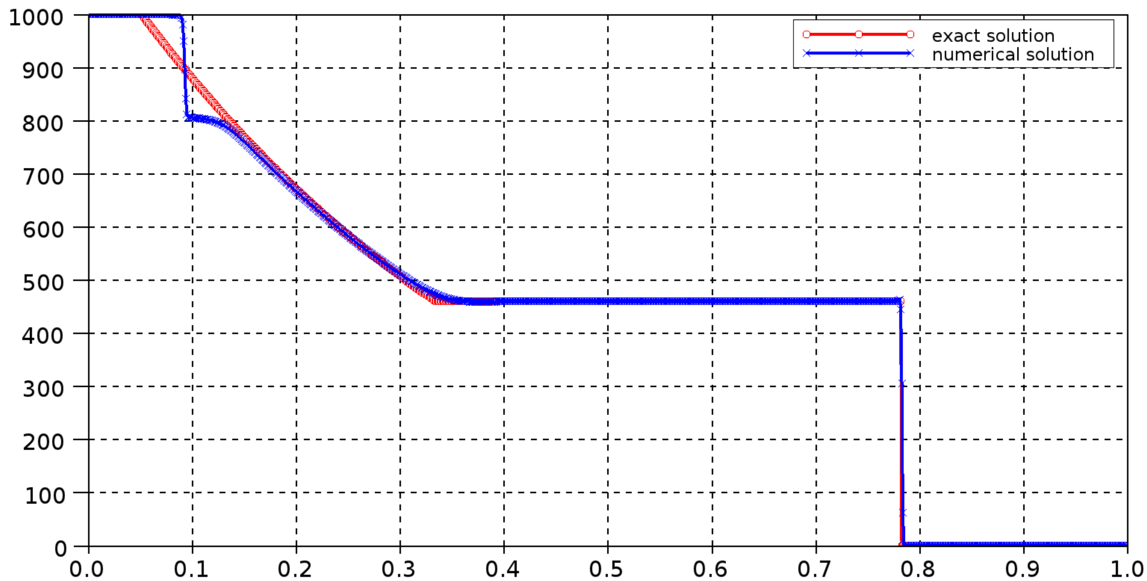


Figure 3.10: Test 3, $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme – $h = 0.001$ and $\delta t = h/100$ – Pressure at $t = 0.012$.

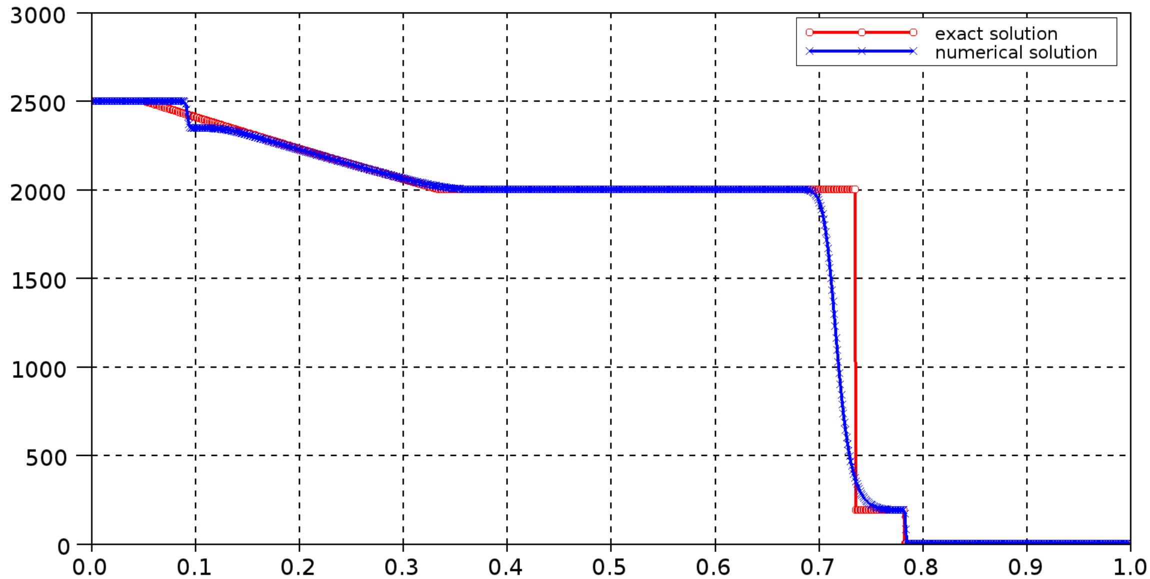


Figure 3.11: Test 3, $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme – $h = 0.001$ and $\delta t = h/100$ – Internal energy at $t = 0.012$.

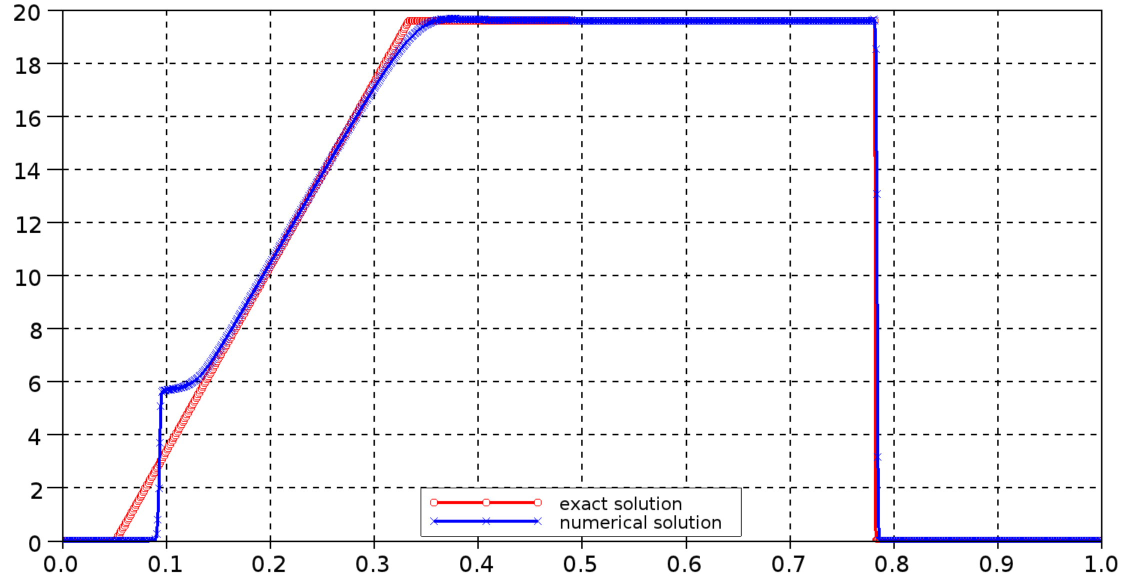


Figure 3.12: Test 3, $\rho \rightsquigarrow u \rightsquigarrow e \rightsquigarrow p$ scheme – $h = 0.001$ and $\delta t = h/100$ – Velocity at $t = 0.012$.

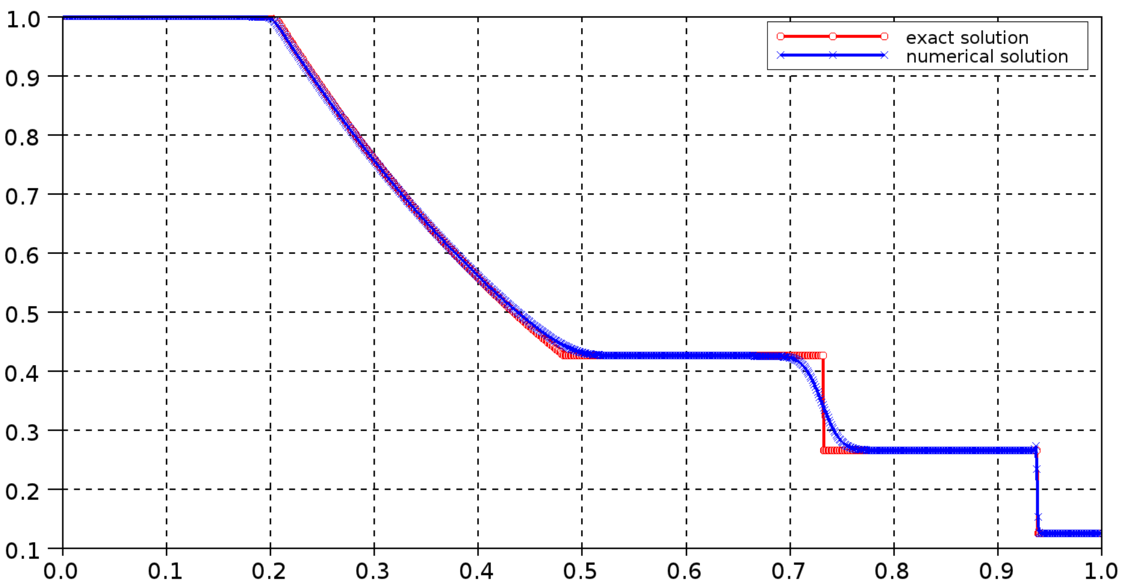


Figure 3.13: Test 1 – $h = 0.001$ and $\delta t = h/6$ – Density at $t = 0.25$.

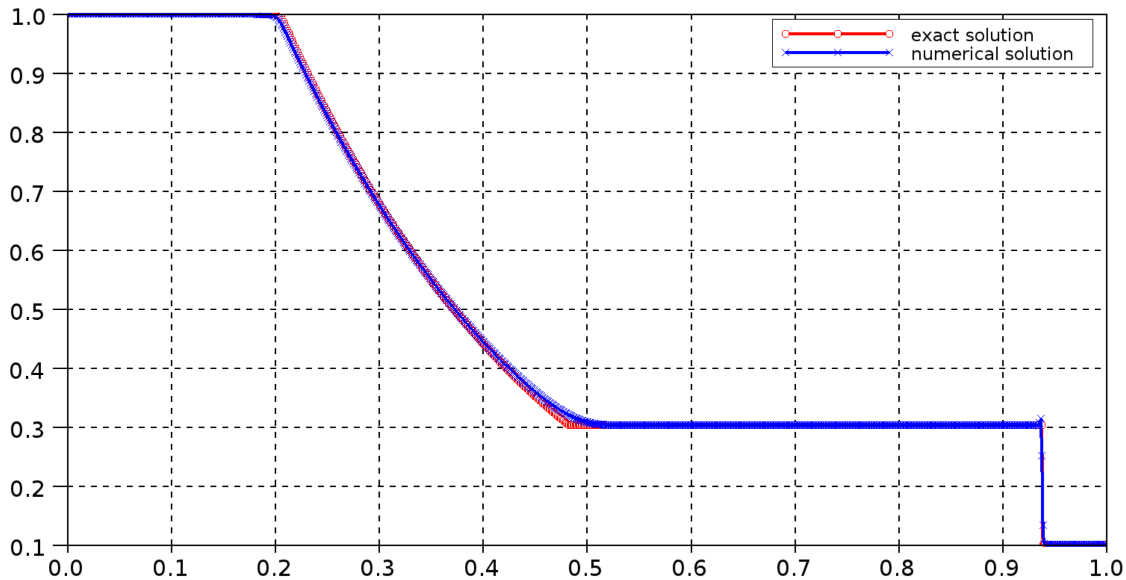


Figure 3.14: Test 1 – $h = 0.001$ and $\delta t = h/6$ – Pressure at $t = 0.25$.

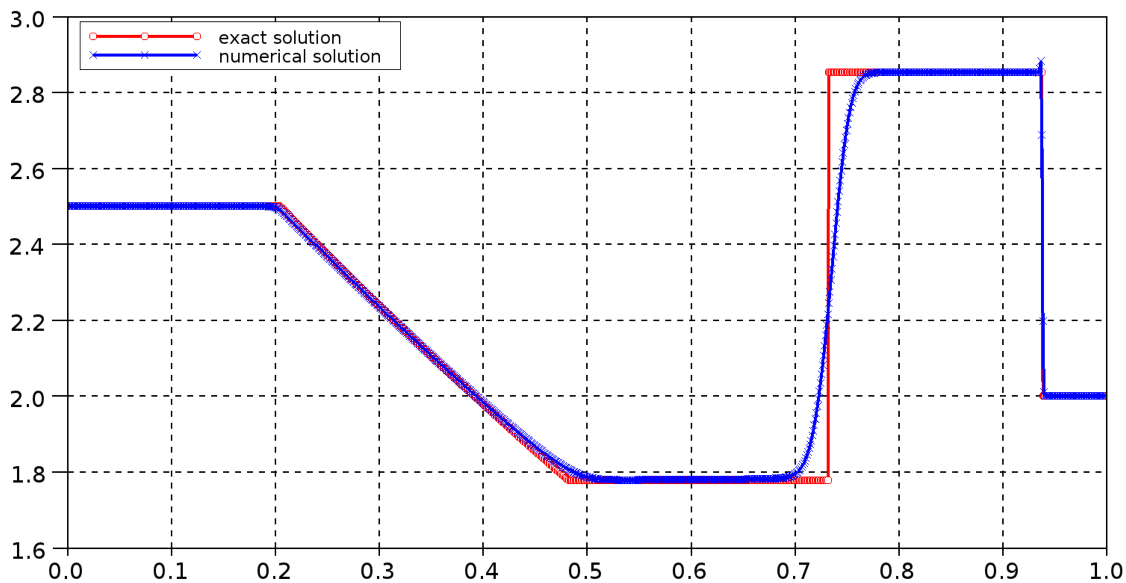


Figure 3.15: Test 1 – $h = 0.001$ and $\delta t = h/6$ – Internal energy at $t = 0.25$.

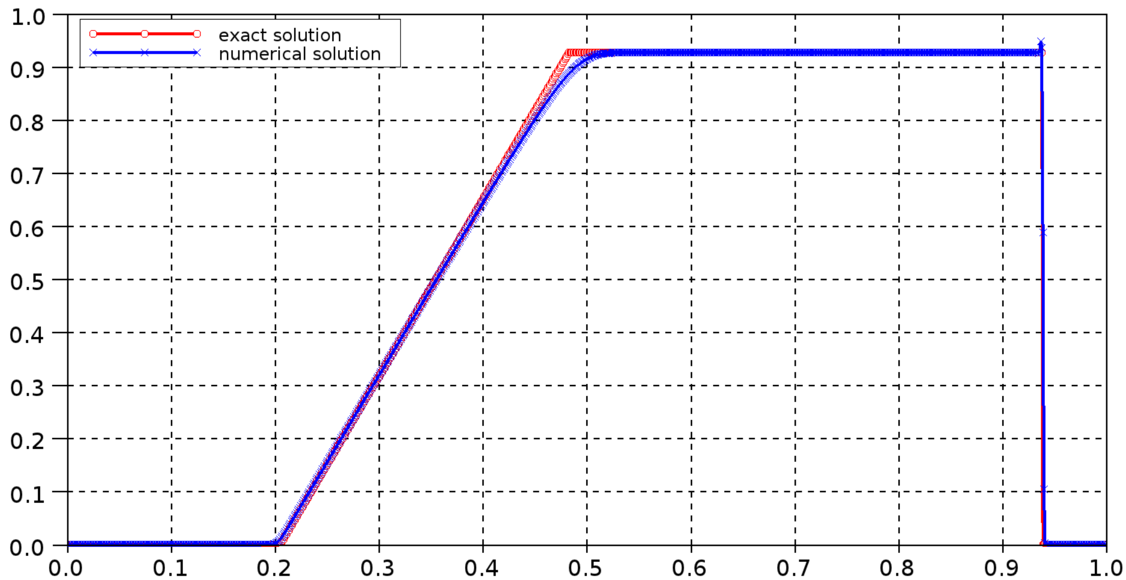


Figure 3.16: Test 1 – $h = 0.001$ and $\delta t = h/6$ – Velocity at $t = 0.25$.

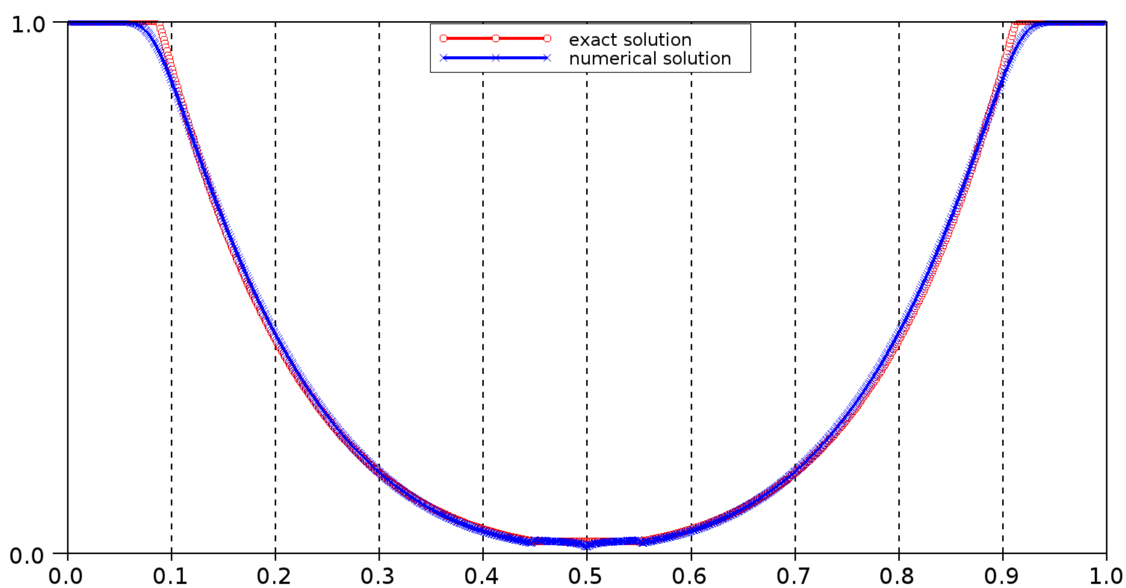


Figure 3.17: Test 2 – $h = 0.001$ and $\delta t = h/5$ – Density at $t = 0.15$.

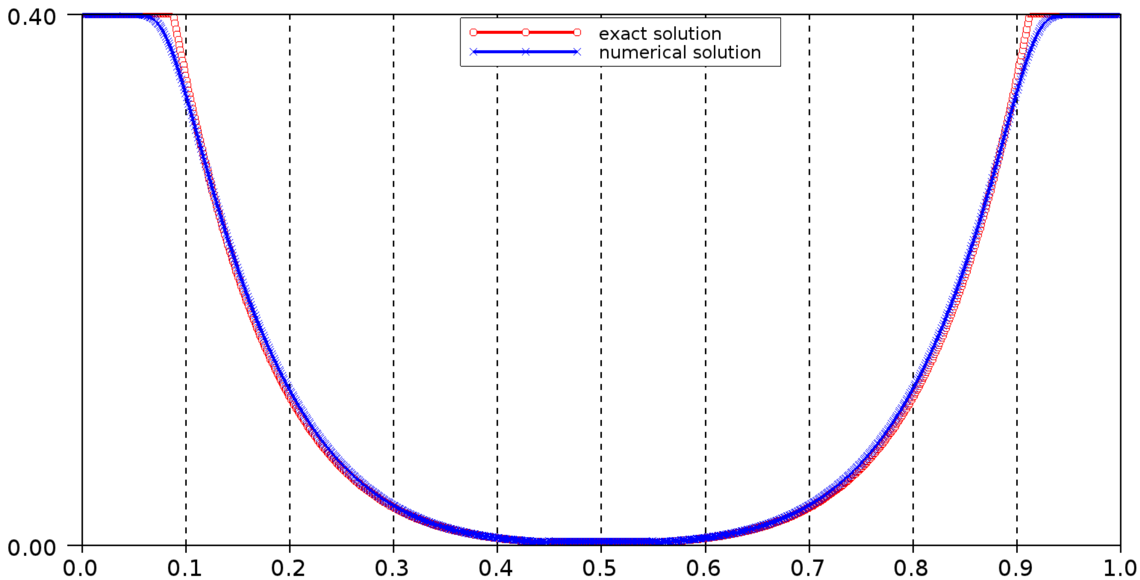


Figure 3.18: Test 2 – $h = 0.001$ and $\delta t = h/5$ – Pressure at $t = 0.15$.

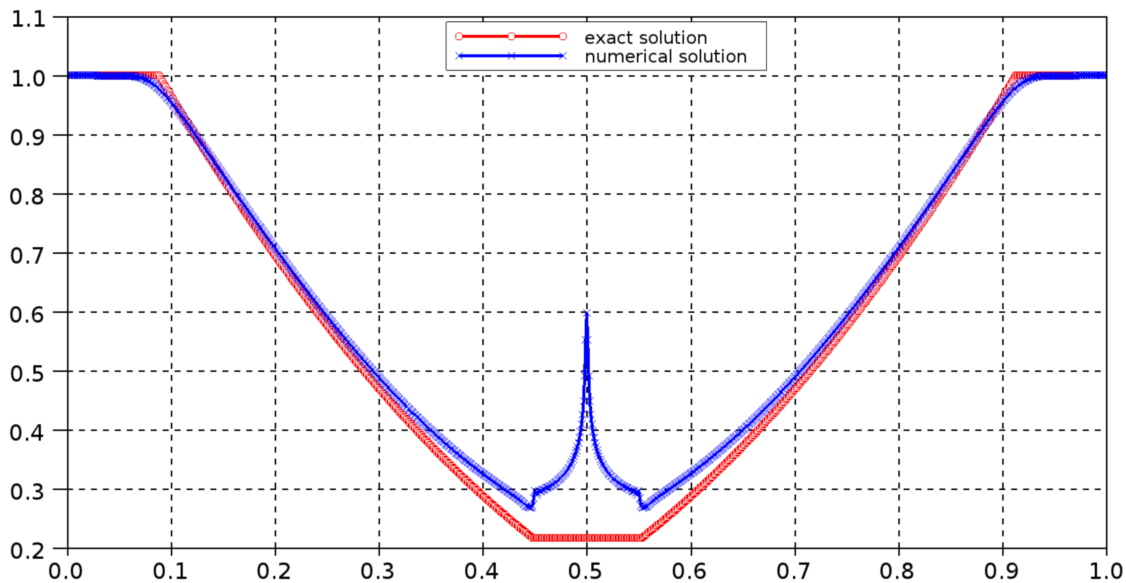


Figure 3.19: Test 2 – $h = 0.001$ and $\delta t = h/5$ – Internal energy at $t = 0.15$.

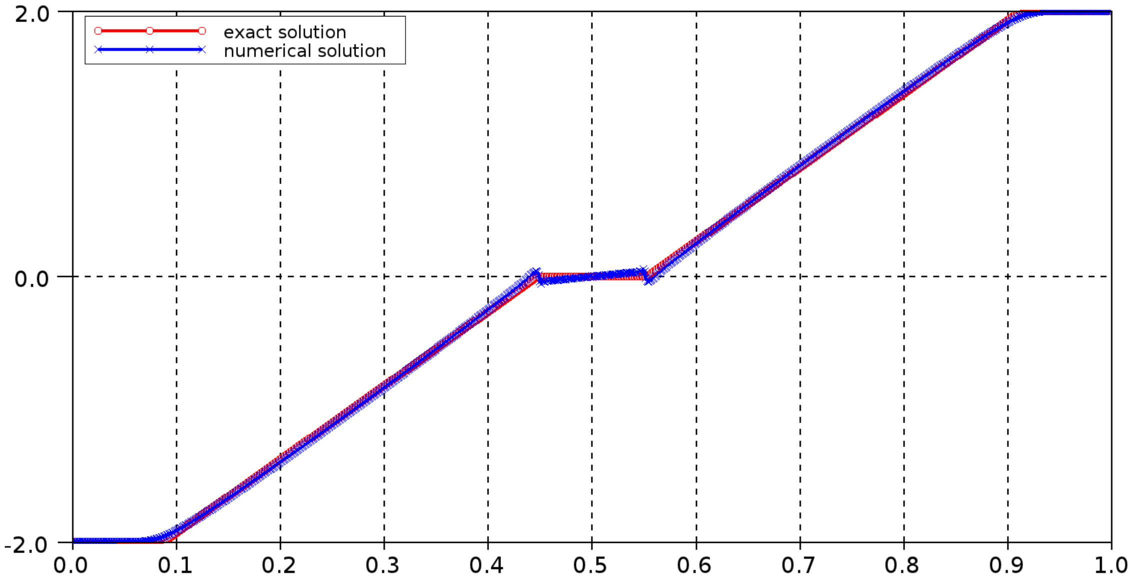


Figure 3.20: Test 2 – $h = 0.001$ and $\delta t = h/5$ – Velocity at $t = 0.15$.

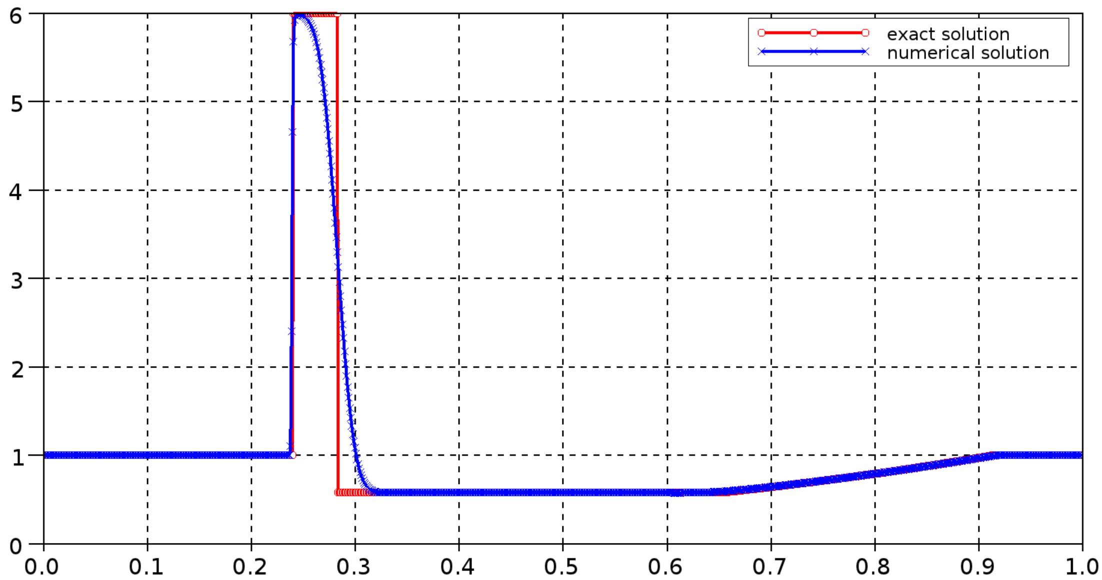


Figure 3.21: Test 4 – $h = 0.001$ and $\delta t = h/30$ – Density at $t = 0.035$.

The chosen left and right states are given by:

$$\text{left state:} \begin{bmatrix} \rho_L = 1 \\ u_L = 0 \\ p_L = 0.01 \end{bmatrix}; \qquad \text{right state:} \begin{bmatrix} \rho_R = 1 \\ u_R = 0 \\ p_R = 100 \end{bmatrix}.$$

The computational domain is $\Omega = (0, 1)$ and the final time is $T = 0.035$. The left wave is a shock and the right one is a rarefaction wave.

Results obtained with $h = 0.001$ and $\delta t = h/30$ at $t = T$ are shown on Figures 3.21–3.24.

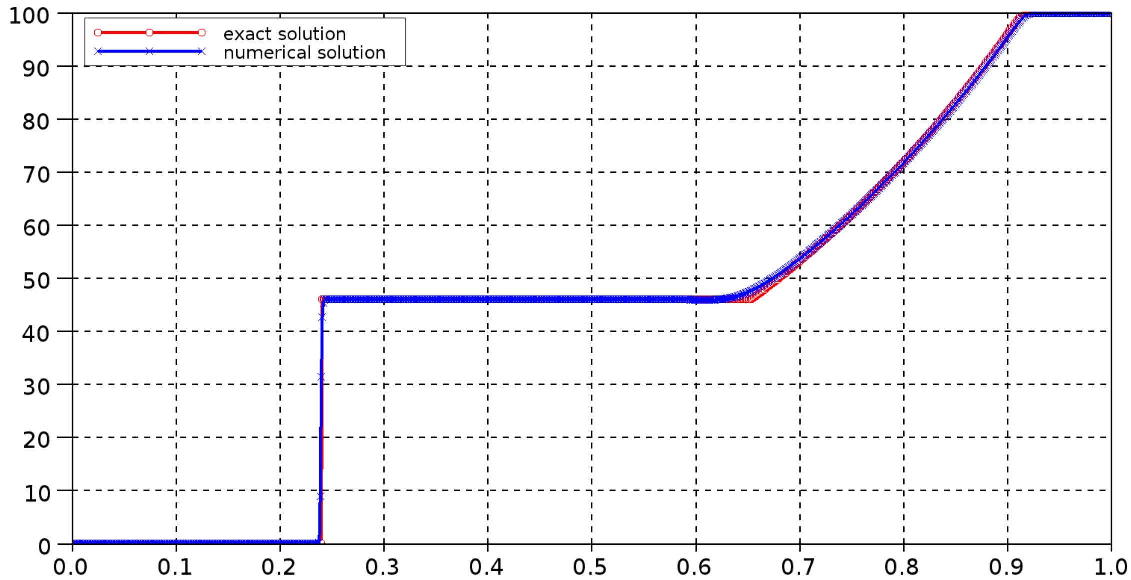


Figure 3.22: Test 4 – $h = 0.001$ and $\delta t = h/30$ – Pressure at $t = 0.035$.

3.5.5 Test 5

The chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 5.99924 \\ u_L = 19.5975 \\ p_L = 460.894 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 5.99242 \\ u_R = -6.19633 \\ p_R = 46.0950 \end{bmatrix}.$$

The computational domain is $\Omega = (0, 1)$ and the final time is $T = 0.035$. Both left and right waves are shocks.

Results obtained with $h = 0.001$ and $\delta t = h/40$ at $t = T$ are shown on Figures 3.17–3.20.

3.6 Conclusion

We have presented in this chapter an explicit scheme based on staggered meshes for Euler equations. This algorithm uses a very simple first-order upwinding strategy which

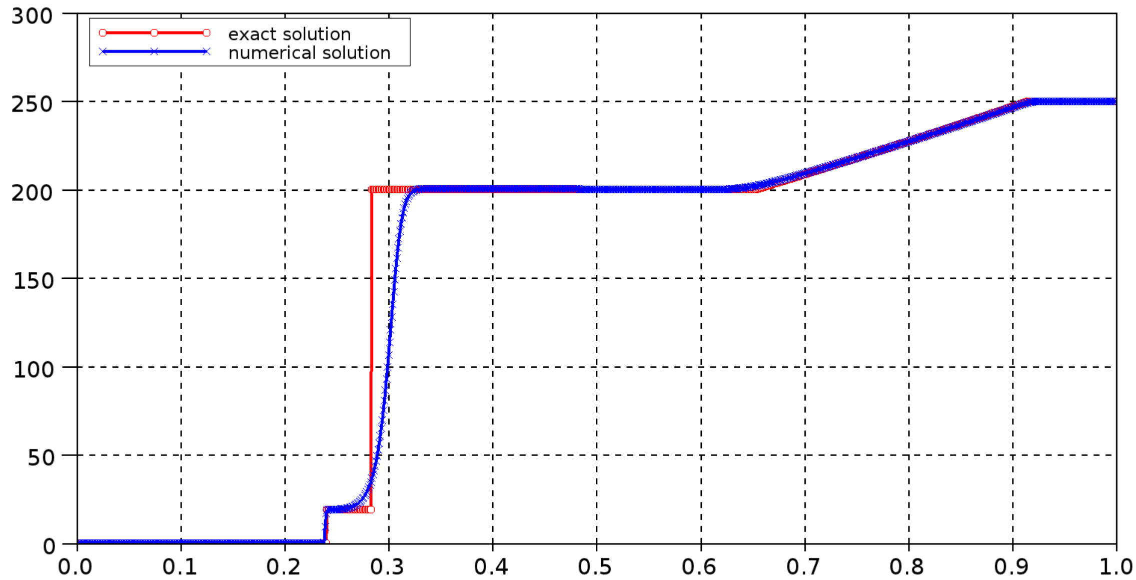


Figure 3.23: Test 4 – $h = 0.001$ and $\delta t = h/30$ – Internal energy at $t = 0.035$.

consists, equation by equation, to implement an upwind discretization with respect of the material velocity of the convection term. In addition, it solves the internal energy balance instead of the total energy balance, and thus turns out to be non-conservative: indeed, the total energy conservation law is only recovered at the limit of vanishing time and space steps, thanks to the addition of corrective source terms in the discrete internal energy balance. Under CFL-like conditions based on the material velocity only (by opposition to the celerity of waves), this scheme preserves the positivity of the density, the internal energy and the pressure (in other words, the scheme preserves the convex of admissible states), and its solution satisfies a property of conservation (in fact, as often at the discrete level, non-increase) of the integral of the total energy over the computational domain. Finally, the scheme has been shown to be consistent for 1D problems, in the sense that, if a sequence of numerical solutions obtained with more and more refined meshes (and, accordingly, smaller and smaller time steps) converges, then the limit is a weak solution to the continuous problem.

This theoretical result may probably be extended in two directions: first, to check whether limits of convergent sequences are entropy solutions, and, second, to deal with

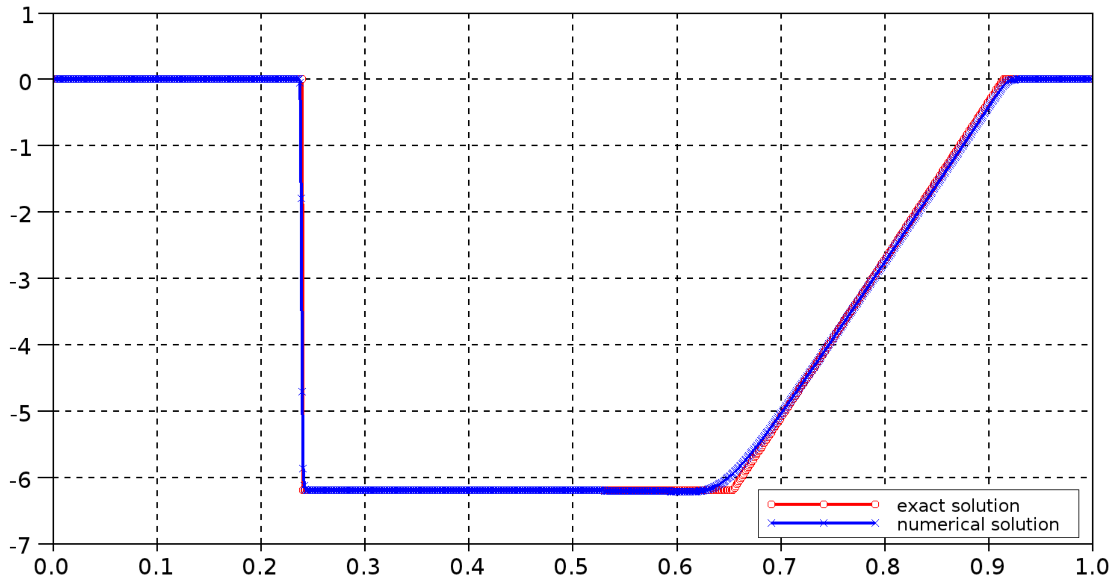


Figure 3.24: Test 4 – $h = 0.001$ and $\delta t = h/30$ – Velocity at $t = 0.035$.

the consistency issue in the multi-dimensional case. The investigation of this latter point should help to clarify the constraints on mesh generality imposed by consistency requirements, in particular with the aim to design a discretization able to cope with non-conform locally refined meshes. This work is now being undertaken.

Numerical studies show that the proposed algorithm is stable, even if the largest time step before blow-up is smaller than suggested by the above-mentioned CFL conditions. This behaviour had to be expected, since these CFL conditions only involve the velocity (and not the celerity of the acoustic waves): indeed, were they the only limitation, we would have obtained an explicit scheme stable up to the incompressible limit. However, the mechanisms leading to the blow-up of the scheme (or, conversely, the way to fix the time step to ensure stability) remain to be understood. In addition, still as expected, the scheme is rather diffusive, especially at contact discontinuities; MUSCL-like extensions are under development to cure this problem, possibly combined with a strategy similar to the so-called entropy-viscosity technique [21, 22] to damp spurious oscillations which are sometimes observed when the velocity is small (refer Chapter 2, Section 2.5 for a numerical study of this issue).

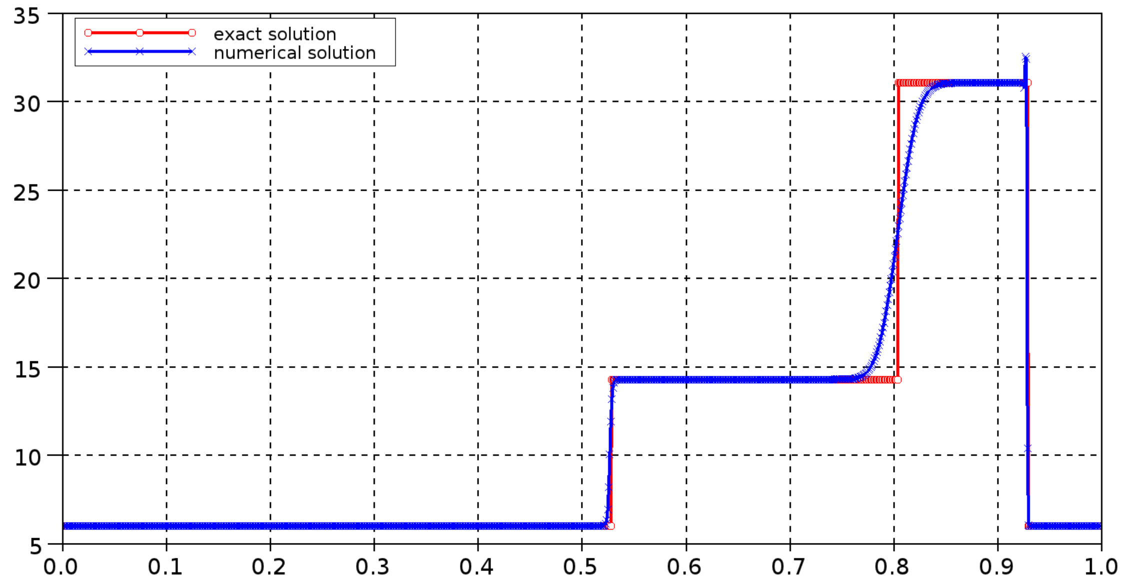


Figure 3.25: Test 5 – $h = 0.001$ and $\delta t = h/5$ – Density at $t = 0.035$.

Since the proposed scheme uses very simple numerical fluxes, it is well suited to large multi-dimensional parallel computing applications, and such studies are now beginning at IRSN. Still for the same reasons (and, in particular, because the construction of the discretization does not require the solution of the Riemann problem), it seems that the presented approach offers natural extensions to more complex problems, such as reacting flows; this development is foreseen at IRSN, for applications to explosion hazards.

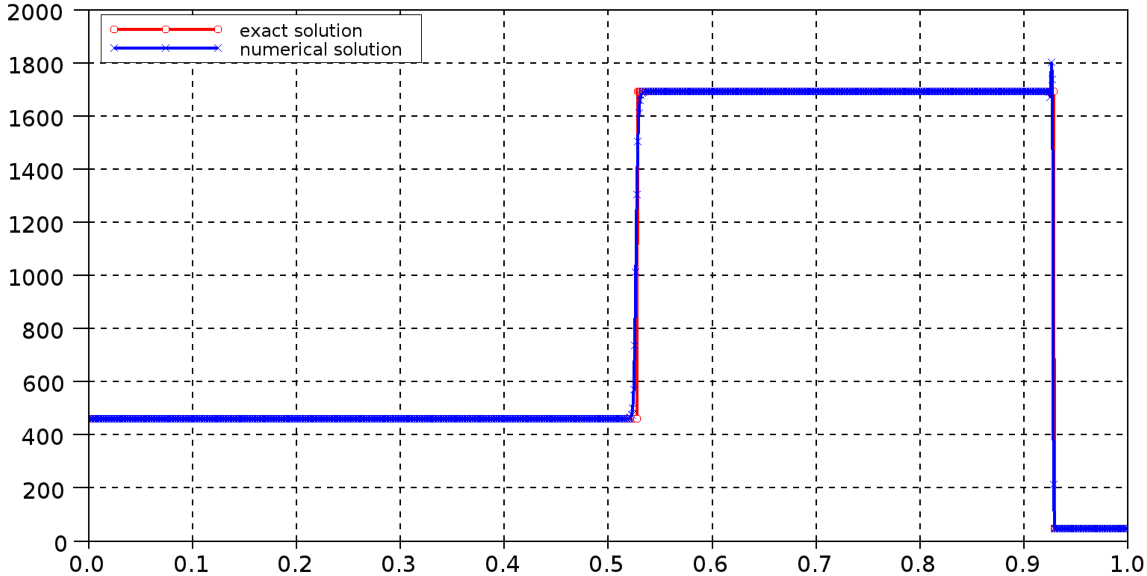


Figure 3.26: Test 5 – $h = 0.001$ and $\delta t = h/5$ – Pressure at $t = 0.035$.

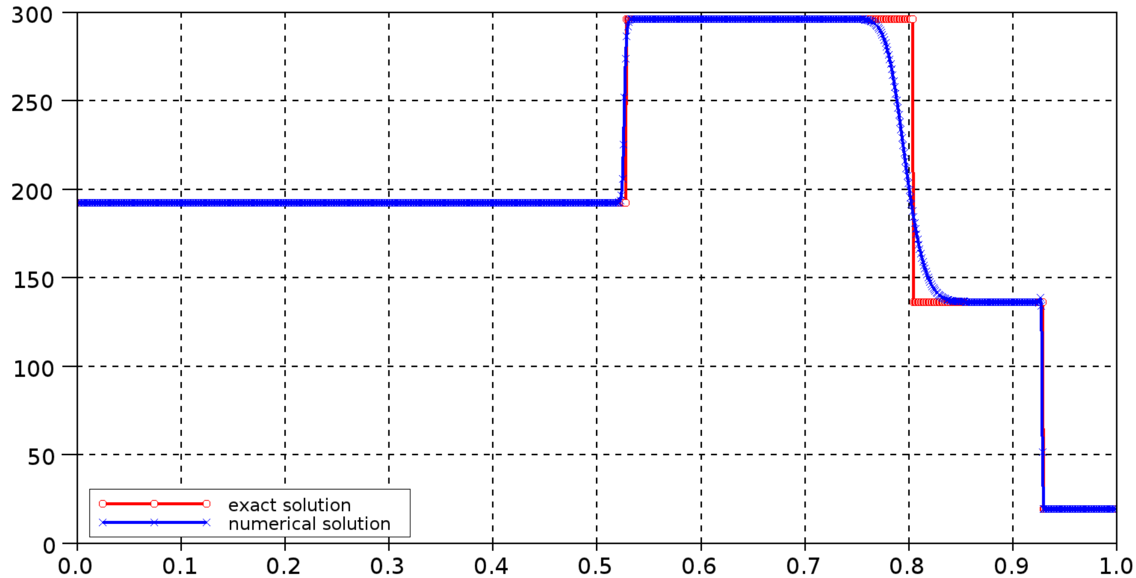


Figure 3.27: Test 5 – $h = 0.001$ and $\delta t = h/5$ – Internal energy at $t = 0.035$.

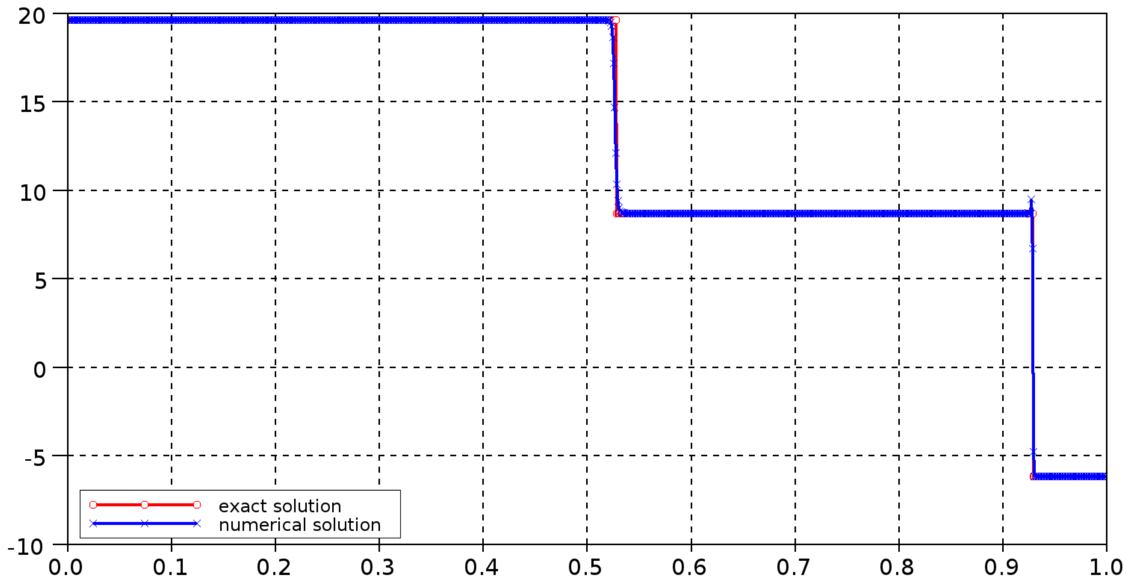


Figure 3.28: Test 5 – $h = 0.001$ and $\delta t = h/5$ – Velocity at $t = 0.035$.

Chapter 4

Radial compressible flows

4.1 Introduction

In the first two chapters, we studied numerical schemes for the (barotropic) Euler equations in case of irrotational flows. However, there are situations for which blast waves propagate in radial and spherical trajectories for two and three-dimensional flows, respectively, such as the propagation or explosion in a porous medium. This motivates the development and/or modification of existing schemes for the discretization of the non-conservative systems of equations which reads, for the barotropic Euler equations

$$\partial_t \rho + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u) = 0 \tag{4.1a}$$

$$\partial_t (\rho u) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u^2) + \partial_r p = 0 \tag{4.1b}$$

$$p = \wp(\rho) = \rho^\gamma \tag{4.1c}$$

where r is the radial direction, t is time, ρ, u and p are the density, radial velocity and pressure in the flow, and $\gamma \geq 1$ is a coefficient specific to the considered fluid. The parameter α depends on the space dimension d: $\alpha = d - 1$. For $\alpha = 0$, we reproduce the one-dimensional flow which was surveyed in Chapter 2 and 3. The cases $\alpha = 1$ and $\alpha = 2$ are corresponding to the two and three-dimensional problems in cylindrical and spherical symmetry coordinates, respectively. The problem is supposed to be posed over $\Omega \times (0, T)$, where $\Omega = [0, +\infty)$ and $(0, T)$ is a finite time interval. This system must be supplemented by initial conditions for ρ and u, denoted by ρ_0 and u_0, and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition where the radial velocity vanishes at any time on $\partial\Omega$.

A weak solution to the continuous problem (4.1) satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\,\partial_t\varphi + \rho\,u\,\partial_r\varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,r^\alpha\,\mathrm{d}r = 0, \tag{4.2a}$$

$$\begin{aligned} -\int_0^T \int_\Omega \Big(\big[\rho\,u\,\partial_t\varphi + (\rho\,u^2 + p)\,\partial_r\varphi\big]\, r^\alpha + p\,\partial_r(r^\alpha\varphi)\Big)\,\mathrm{d}r\,\mathrm{d}t \\ - \int_\Omega \rho_0(x)\,u_0(x)\,\varphi(x,0)\,r^\alpha\,\mathrm{d}r = 0, \end{aligned} \tag{4.2b}$$

$$p = \rho^\gamma. \tag{4.2c}$$

Let us denote by E_k the kinetic energy $E_k = \frac{1}{2}\,u^2$. Taking the product of (4.1b) by u yields, after formal compositions of partial derivatives and using the mass balance (4.1a):

$$\rho\,u\,\partial_t u + \rho\,u^2\,\partial_r u + u\,\partial_r p = 0.$$

Invoking one more time the mass balance, we obtain the kinetic energy equation

$$\partial_t(\rho E_k) + \frac{1}{r^\alpha}\partial_r\big(r^\alpha \rho\,E_k\,u\big) + u\,\partial_r p = 0. \tag{4.3}$$

Let us now define the function $\mathcal{P}$, from $(0,+\infty)$ to $\mathbb{R}$, as a primitive of $s \mapsto \wp(s)/s^2$; this quantity is often called the elastic potential. Let $\mathcal{H}$ be the function defined by $\mathcal{H}(s) = s\mathcal{P}(s),\ \forall s \in (0,+\infty)$. For the specific equation of state $\wp$ used here, we obtain:

$$\mathcal{H}(s) = s\,\mathcal{P}(s) = \begin{cases} \dfrac{s^\gamma}{\gamma - 1} & \text{if } \gamma > 1, \\[2ex] s\ln(s) & \text{if } \gamma = 1. \end{cases} \tag{4.4}$$

Since $\wp$ is an increasing function, $\mathcal{H}$ is convex. In addition, it may easily be checked that $\rho\mathcal{H}'(\rho) - \mathcal{H}(\rho) = \wp(\rho)$. Therefore, by a formal computation, detailed in the appendix, multiplying (4.1a) by $\mathcal{H}'(\rho)$ yields:

$$\partial_t\big(\mathcal{H}(\rho)\big) + \frac{1}{r^\alpha}\partial_r\big(r^\alpha\mathcal{H}(\rho)\,u\big) + \frac{1}{r^\alpha}p\,\partial_r(r^\alpha u) = 0. \tag{4.5}$$

Let us denote by $\mathcal{S}$ the quantity $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$. Summing (4.3) and (4.5), we get:

$$\partial_t \mathcal{S} + \frac{1}{r^\alpha} \partial_r \big(r^\alpha (\mathcal{S} + p)\, u \big) = 0. \tag{4.6}$$

In fact, to avoid invoking unrealistic regularity assumptions, such a computation should be done on regularized equations (obtained by adding diffusion perturbation terms); when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (4.6), so that we get in the distribution sense:

$$\partial_t \mathcal{S} + \frac{1}{r^\alpha} \partial_r \big(r^\alpha (\mathcal{S} + p)\, u \big) \leq 0. \tag{4.7}$$

The quantity $\mathcal{S}$ is an entropy of the system, and an entropy solution to (4.1) is thus required to satisfy:

$$\begin{aligned} &\forall \varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big),\ \varphi \geq 0, \\ &\qquad \int_0^T \int_\Omega \big[-\mathcal{S} \partial_t \varphi - (\mathcal{S} + p)\, u\, \partial_r \varphi \big]\, r^\alpha \, \mathrm{d}r \, \mathrm{d}t - \int_\Omega \mathcal{S}_0\, \varphi(r,0)\, r^\alpha \, \mathrm{d}r \leq 0, \end{aligned} \tag{4.8}$$

with $\mathcal{S}_0 = \frac{1}{2} \rho_0 u_0^2 + \mathcal{H}(\rho_0)$. Then, since the radial velocity is prescribed to zero at the boundary, integrating (4.7) over Ω yields:

$$\frac{d}{dt} \int_\Omega \big[\frac{1}{2}\, \rho\, u^2 + \mathcal{H}(\rho) \big]\, r^\alpha \, \mathrm{d}r \leq 0. \tag{4.9}$$

Since $\rho \geq 0$ by (4.1a) (and the associated initial and boundary conditions) and the function $s \mapsto \mathcal{H}(s)$ is bounded by below and increasing at least for s large enough, Inequality (4.9) provides an estimate on the solution.

Let us now turn to the Euler equations on cylindrical and spherical coordinate systems under the non-conservative form:

$$\partial_t \rho + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u) = 0 \tag{4.10a}$$

$$\partial_t (\rho u) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho u^2) + \partial_r p = 0 \tag{4.10b}$$

$$\partial_t (\rho E) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho E u) + \frac{1}{r^\alpha} \partial_r (r^\alpha p u) = 0 \tag{4.10c}$$

$$E = \frac{1}{2} u^2 + e \tag{4.10d}$$

$$p = (\gamma - 1) \rho e \tag{4.10e}$$

where E and e stand for the total and internal energy respectively, and $\gamma > 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0, T)$, where $\Omega = [0; +\infty)$ and $(0, T)$ is a finite time interval. Substracting the relation (4.3) from the total energy balance (4.10c), we obtain the internal energy balance equation:

$$\partial_t (\rho e) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho e u) + \frac{1}{r^\alpha}\, p\, \partial_r (r^\alpha u) = 0. \tag{4.11}$$

Since,

- thanks to the mass balance equation, the first two terms in the left-hand side of (4.11) may be recast as a transport operator: $\partial_t(\rho e) + \frac{1}{r^\alpha} \partial_r (r^\alpha \rho e u) = \rho\, [\partial_t e + u\, \partial_r e]$,
- and, from the equation of state, the pressure vanishes when $e = 0$,

this equation implies, if $e \geq 0$ at $t = 0$ and with suitable boundary conditions, that e remains non-negative at all times.

A weak solution to the continuous problem (4.10) satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times$

$[0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\, \partial_t \varphi + \rho\, u\, \partial_r \varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \tag{4.12a}$$

$$\begin{aligned} -\int_0^T \int_\Omega \Big[\rho\, u\, \partial_t \varphi + (\rho\, u^2 + p)\, \partial_r \varphi\Big]\, r^\alpha + p\, \partial_r (r^\alpha \varphi) \,\mathrm{d}r\,\mathrm{d}t \\ - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \end{aligned} \tag{4.12b}$$

$$-\int_0^T \int_\Omega \Big[\rho\, E\, \partial_t \varphi + (\rho\, E + p)\, u\, \partial_r \varphi\Big]\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\, E_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r = 0, \tag{4.12c}$$

$$p = (\gamma - 1)\rho\, e, \qquad E = \frac{1}{2} u^2 + e, \qquad E_0 = \frac{1}{2} u_0^2 + e_0. \tag{4.12d}$$

Note that relations (4.2) and (4.12) are not sufficient to define a weak solution to the problem (4.1) and (4.10), respectively, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks* (*i.e.* shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions).

This chapter gives, in the case of the above equations in cylindrical and spherical coordinates, an explicit variant of an all-Mach-number pressure correction scheme [15, 26] which has been studying in the framework of the simulation of compressible flows and implementing in the industrial computer code ISIS [33]. The initial motivation of ISIS was to provide in the same software an efficient alternative for quickly varying unstationary flows, with a characteristic Mach number in the range or greater than the unity.

We use a staggered finite volume or finite element discretization in space. For the sake of stability, the upwinding technique is applied equation-by-equation with respect to the material velocity only which is contrary to the Riemann solvers for hyperbolic systems, where upwinding is performed based on the celerity of waves. The pressure gradient is defined as the transpose of the natural velocity divergence, and is thus centered. Last but not least, the velocity convection term is built is such a way to allow to derive a discrete kinetic energy balance.

We prove for the scheme(s) the following results:

- A discrete kinetic energy balance with some residual (*i.e.* a discrete analogue of (4.3)) on dual cells.
- A discrete elastic potential equation with some rest terms (*i.e.* a discrete analogue of (4.5)) on primal cells for the barotropic Euler equations. These rest terms, naturally arising from computations at the discrete level, are controlled by a CFL condition to obtain the discrete version of entropy condition (4.8)
- Discrete internal energy balances with some residual (*i.e.* a discrete analogue of (4.11)) on primal cells for the Euler equations. In the contrary to rest terms in the elastic potential equation, the residual here are imposed to complement rest terms in the discrete kinetic energy balance at the limit, when the mesh size and time step tend to zero, in order to recover the total energy equation.
- Finally, passing to the limit in all equations and supposing the convergence of scheme(s), the limits are shown to be weak solutions of the continuous problem(s), and thus to satisfy the Rankine-Hugoniot conditions. In particular, they are entropy solutions to the barotropic Euler equations.

This chapter is structured as follows. We begin with the presentation of the space discretization (Section 4.2). The next section is dedicated to the barotropic Euler equations (Section 4.3). In this section, we have three subsections including the scheme description in Subsection 4.3.1. The construction of discrete kinetic energy and elastic potential equations are described in Subsection 4.3.2. The consistency of the scheme can be found in Subsection 4.3.3. The structure for the section of Euler equation (Section 4.4) is the same as the barotropic Euler equations except the elastic potential balance is replaced by corrective source terms in the internal energy equation (4.4.2). The discrete kinetic energy and elastic potential balances are obtained as particular cases of more general results applying to the explicit finite volume discretization of transport operators, which are established in Chapter 2, Appendix 2.7.1. Finally, we present some numerical tests to assess the behaviour of the algorithms (Section 4.5).

4.2 Meshes and unknowns

For any $K \in \mathcal{M}$, we denote by h_K its length (so $h_K = |K|$); when we write $K = [\sigma\sigma']$, this means that either $K = (x_\sigma, x_{\sigma'})$ or $K = (x_{\sigma'}, x_\sigma)$; if we need to specify the order, *i.e.* $K = (x_\sigma, x_{\sigma'})$ with $x_\sigma < x_{\sigma'}$, then we write $K = [\overrightarrow{\sigma\sigma'}]$. For an interface $\sigma = K|L$ between two cells K and L, we define $h_\sigma = (h_K + h_L)/2$, so, by definition of the dual mesh, $h_\sigma = |D_\sigma|$. If we need to specify the order of the cells K and L, say K is left of L, then we write $\sigma = \overrightarrow{K|L}$.

The volume of K denoted by $|V_K|$ reads

$$|V_K| = \frac{r_{\sigma'}^{\alpha+1} - r_\sigma^{\alpha+1}}{\alpha+1}, \qquad \forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \tag{4.13}$$

while the volume of D_σ denoted by $|V_\sigma|$ can be selected based on the way we define the dual radius r_σ. In the spirit of ISIS, the mean value of volumes of two primal cells K and L gives the volume of the dual cell D_σ

$$|V_\sigma| = \frac{|V_K| + |V_L|}{2}, \qquad \forall \sigma = K|L \in \mathcal{E}_{\rm int}. \tag{4.14}$$

In this way, the primal radius r_K reads

$$r_K = \sqrt[\alpha+1]{\frac{r_\sigma^{\alpha+1} + r_{\sigma'}^{\alpha+1}}{2}}, \qquad \forall K = [\sigma\sigma'] \in \mathcal{M}. \tag{4.15}$$

Otherwise, given $r_K = (r_\sigma + r_{\sigma'})/2$, $\forall K \in \mathcal{M}$, we define the volume of D_σ as the integral on $[r_K, r_L]$

$$|V_\sigma| = \frac{r_L^{\alpha+1} - r_K^{\alpha+1}}{\alpha+1}, \qquad \forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}. \tag{4.16}$$

The volume of $K \cap D_\sigma$ denoted by $|V_{K,\sigma}|$, in both choices of $|V_\sigma|$, is given by

$$|V_{K,\sigma}| = \frac{|V_K|}{2}, \qquad \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}. \tag{4.17}$$

Both definitions for the volumes of dual cells, in fact, gives the same numerical solution, up to a very small tolerance, when mesh size and time step tend to zero. Therefore, in this chapter, we work only with the mean value volume case.

Let a sequence of discretizations $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ be given. We define the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ by $h^{(m)} = \sup_{K\in\mathcal{M}^{(m)}} h_K$. Let $\rho^{(m)}$, $p^{(m)}$, $e^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (4.42) with the mesh $\mathcal{M}^{(m)}$ and the time step $\delta t^{(m)}$. For a fixed m, the unknowns in our discretizations are constant on the mesh, *i.e.* on primal cells, ρ, p, e are constant and u is constant on dual cells. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho^{(m)}$, the pressure $p^{(m)}$, the internal energy $e^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0,T)$ by:

$$\rho^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (\rho^{(m)})^n_K\, \mathcal{X}_K(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{4.18a}$$

$$u^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} (u^{(m)})^n_\sigma\, \mathcal{X}_{D_\sigma}(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{4.18b}$$

$$p^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (p^{(m)})^n_K\, \mathcal{X}_K(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{4.18c}$$

$$e^{(m)}(x,t) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (e^{(m)})^n_K\, \mathcal{X}_K(x)\, \mathcal{X}_{(n,n+1]}(t), \tag{4.18d}$$

where $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{(n,n+1]}$ stand for the characteristic function of the intervals K, D_σ and $(t^n, t^{n+1}]$ respectively.

For discrete functions q and v defined on the primal and dual mesh, respectively, we define a discrete $\mathrm{L}^1((0,T);\mathrm{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |q^n_L - q^n_K|, \qquad \|v\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\tilde{\mathcal{E}}_{\mathrm{int}}} |v^n_{\sigma'} - v^n_\sigma|,$$

and a discrete $\mathrm{L}^1(\Omega;\mathrm{BV}((0,T)))$ norm by:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K\in\mathcal{M}} |V_K| \sum_{n=0}^{N-1} |q^{n+1}_K - q^n_K|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma\in\mathcal{E}} |V_\sigma| \sum_{n=0}^{N-1} |v^{n+1}_\sigma - v^n_\sigma|.$$

4.3 The barotropic Euler equations

4.3.1 The scheme

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|K|} \int_K \rho_0(x) \,\mathrm{d}x, \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x) \,\mathrm{d}x, \end{aligned} \tag{4.19a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M},$$

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{4.19b}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{4.19c}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}},$$

$$\frac{|V_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + r_\sigma^\alpha \,(p_L^{n+1} - p_K^{n+1}) = 0. \tag{4.19d}$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (4.19b) is obtained by the discretization of the mass balance equation (4.1a) over the primal mesh, and F_σ^n stands for the discrete mass flux across σ outward K, which, because of the impermeability condition, vanishes on $\partial\Omega$ and is given on the internal edges by:

$$\forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad F_\sigma^n = r_\sigma^\alpha \rho_\sigma^n u_\sigma^n, \tag{4.20}$$

where the upwind approximation for the density at the edge, ρ_σ^n, is defined by

$$\rho_\sigma^n = \left| \begin{array}{ll} \rho_K^n & \text{if } u_\sigma^n \geq 0, \\ \\ \rho_L^n & \text{otherwise.} \end{array} \right. \tag{4.21}$$

We now turn to the discrete momentum balance (4.19d), which is obtained by discretizing the momentum balance equation (4.1b) on the dual cells associated to the faces of the mesh. For the discretization of the time derivative, we need to provide a definition for the values $\rho_{D_\sigma}^{n+1}$ and $\rho_{D_\sigma}^{n}$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}},\ \text{for } k = n \text{ and } k = n+1, \qquad |V_\sigma|\ \rho_{D_\sigma}^k = |V_{K,\sigma}|\ \rho_K^k + |V_{L,\sigma}|\ \rho_L^k. \tag{4.22}$$

where $|V_{K,\sigma}| = |V_K|/2$, $\forall K \in \mathcal{M}$. The discrete mass flux F_K^n in the discretization of the convection term reads

$$\forall K = [\sigma\sigma'] \in \mathcal{M}, \qquad F_K^n = \frac{1}{2}\,(F_\sigma^n + F_{\sigma'}^n), \tag{4.23}$$

Therefore, we obtain the discrete mass balance equation on dual cells:

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}, \qquad \frac{|V_\sigma|}{\delta t}\,(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^{n}) + F_L^n - F_K^n = 0, \tag{4.24}$$

Let us remark that a dual edge lying on the boundary is then also a primal edge, and the flux across that face is zero. Thanks to the discrete mass flux on dual cells, the approximation of u_K^n is given by the upwinding technique:

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad u_K^n = \left| \begin{array}{ll} u_\sigma^n & \text{if } F_K^n \geq 0, \\ u_{\sigma'}^n & \text{otherwise.} \end{array} \right. \tag{4.25}$$

We denote $(\partial_r p)_\sigma^{n+1}$ and $(\partial_r u)_K^{n+1}$, respectively, the discrete derivatives of pressure at the edge σ and the velocity on primal cell K. The last term in Equation (4.19d) known as the discrete version of pressure derivative on the dual cell D_σ is built as the transpose of velocity derivative on the primal cell K. The natural approximation for the derivative of the velocity on primal cells reads

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad (\partial_r u)_K^{n+1} = \frac{1}{h_K}\big(r_{\sigma'}^\alpha\, u_{\sigma'}^{n+1} - r_\sigma^\alpha\, u_\sigma^{n+1}\big). \tag{4.26}$$

Consequently, the discrete derivative of pressure at the edge σ is given by

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \qquad (\partial_r p)_\sigma^{n+1} = \frac{1}{h_\sigma} r_\sigma^\alpha\,\big(p_L^{n+1} - p_K^{n+1}\big). \tag{4.27}$$

Hence, we obtain the duality relation between derivatives of pressure and velocity:

$$\sum_{K\in\mathcal{M}} h_K\, p_K^{n+1}\, (\partial_r u)_K^{n+1} + \sum_{\sigma\in\mathcal{E}_{\rm int}} h_\sigma\, u_\sigma^{n+1}\, (\partial_r p)_\sigma^{n+1} = 0. \tag{4.28}$$

Note that, because of the impermeability boundary conditions, the discrete pressure derivative is not defined at the external edges.

Finally, the initial approximations for ρ and u are given by the average of the initial conditions ρ_0 and u_0 on the primal and dual cells respectively:

$$\begin{aligned} &\forall K\in\mathcal{M}, \quad \rho_K^0 = \frac{1}{|V_K|}\int_K \rho_0(r)\, r^\alpha\, \mathrm{d}r, \\ &\forall \sigma\in\mathcal{E}_{\rm int}, \quad u_\sigma^0 = \frac{1}{|V_\sigma|}\int_{D_\sigma} u_0(r)\, r^\alpha\, \mathrm{d}r. \end{aligned} \tag{4.29}$$

The following positivity result is a classical consequence of the upwind choice in the mass balance equation.

Lemma 4.3.1 (Positivity of the density)**.** *Let ρ^0 be given by* (4.29)*. Then, since u_0 is assumed to be a positive function, $\rho^0 > 0$ and, under the* CFL *condition:*

$$\delta t \le \frac{|V_K|}{r_{\sigma'}^\alpha\, (u_{\sigma'}^n)^+ + r_\sigma^\alpha\, (u_\sigma^n)^-}, \tag{4.30}$$

the solution to the scheme satisfies $\rho^n > 0$, for $1 \le n \le N$.

4.3.2 Discrete kinetic energy and elastic potential balances

We begin by deriving a discrete kinetic energy balance equation, as was already done for the implicit and fractional time step scheme described in [26]. Equation (4.31) is a discrete analogue of Equation (4.3), with an upwind discretization of the convection term.

Lemma 4.3.2 (Discrete kinetic energy balance)**.**
A solution to the system (4.19) *satisfies the following equality: $\forall n \in \{0,\ldots,N-1\}$,*

$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}$, $K = \overrightarrow{\sigma'|\sigma}$ and $L = \overrightarrow{\sigma|\sigma''}$,

$$\frac{1}{2}\,\frac{|V_\sigma|}{\delta t}\Big[\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1})^2 - \rho_{D_\sigma}^{n}(u_\sigma^{n})^2\Big] + \frac{1}{2}\,\big[F_L^n\,(u_L^n)^2 - F_K^n\,(u_K^n)^2\big] + |V_\sigma|\,(\partial_r p)_\sigma^{n+1}\,u_\sigma^{n+1} = -R_\sigma^{n+1}, \quad (4.31)$$

with:

$$R_\sigma^{n+1} = \frac{1}{2}\frac{|V_\sigma|}{\delta t}\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1} - u_\sigma^n)^2 + \frac{1}{2}\Big[(F_L^n)^-(u_{\sigma''}^n - u_\sigma^n)^2 + (F_K^n)^-(u_{\sigma'}^n - u_\sigma^n)^2\Big] \\ - (F_L^n)^-(u_{\sigma''}^n - u_\sigma^n)\,(u_\sigma^{n+1} - u_\sigma^n) - (F_K^n)^+(u_{\sigma'}^n - u_\sigma^n)\,(u_\sigma^{n+1} - u_\sigma^n), \quad (4.32)$$

where, for $a \in \mathbb{R}$, $a^- \geq 0$ is defined by $a^- = -\min(a, 0)$. This remainder term is non-negative under the following CFL condition:

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \qquad \delta t \leq \frac{|V_\sigma|\,\rho_{D_\sigma}^{n+1}}{(F_L^n)^- + (F_K^n)^+}. \quad (4.33)$$

Proof. The proof of this lemma is obtained in the similar way to Lemma 2.3.1 of Chapter 2. □

Similarly, the solution to the scheme (4.19) satisfies a discrete version of the elastic potential identity (4.5), which we now state.

Lemma 4.3.3 (Discrete potential balance)**.** *Let $\mathcal{H}$ be defined by* (4.4)*. A solution to the system* (4.19) *satisfies the following equality, for $K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}$, $\sigma = \overrightarrow{P|K}$, $\sigma' = \overrightarrow{K|Q}$ and $0 \leq n \leq N-1$:*

$$\frac{|V_K|}{\delta t}\,\Big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\Big] + r_{\sigma'}^\alpha\,\mathcal{H}(\rho_{\sigma'}^n)\,u_{\sigma'}^n - r_\sigma^\alpha\,\mathcal{H}(\rho_\sigma^n)\,u_\sigma^n + |V_K|\,p_K^n(\partial_r u^n)_K = -R_K^{n+1}. \quad (4.34)$$

In this relation, the remainder term is defined by:

$$R_K^{n+1} = \frac{1}{2}\frac{|V_K|}{\delta t}\,\mathcal{H}''(\overline{\rho}_{K,1}^n)\,(\rho_K^{n+1} - \rho_K^n)^2 + \big(r_{\sigma'}^\alpha\,\rho_{\sigma'}^n\,u_{\sigma'}^n - r_\sigma^\alpha\,\rho_\sigma^n\,u_\sigma^n\big)\,(\rho_K^{n+1} - \rho_K^n)\,\mathcal{H}''(\overline{\rho}_{K,2}^n) \\ + \frac{1}{2}\Big[r_{\sigma'}^\alpha\,(u_{\sigma'}^n)^-\,(\rho_Q^n - \rho_K^n)^2\,\mathcal{H}''(\overline{\rho}_{\sigma'}^n) + r_\sigma^\alpha\,(u_\sigma^n)^+\,(\rho_P^n - \rho_K^n)^2\,\mathcal{H}''(\overline{\rho}_\sigma^n)\Big], \quad (4.35)$$

with $\overline{\rho}_{K,1}^n,\ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_P^n]\!]$ and $\overline{\rho}_{\sigma'}^n \in [\![\rho_K^n, \rho_Q^n]\!]$, where, for $a,\ b \in \mathbb{R}$, we denote by $[\![a, b]\!]$ the interval $[\![a, b]\!] = \{\theta a + (1-\theta) b,\ \theta \in [0, 1]\}$.

Proof. The proof of this lemma is obtained in the similar way to Lemma 2.3.2 of Chapter 2. $\square$

Unfortunately, it does not seem that $R_K^{n+1} \geq 0$ in any case, and so we are not able to prove a discrete counterpart of the total entropy estimate (4.9), which would yield a stability estimate for the scheme. However, under a condition for a time step which is only slightly more restrictive than a CFL-condition, and under some stability assumptions for the solutions to the scheme, we are able to show that the possible non-positive part of this remainder term tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$, which allows to conclude, in the 1D case, that a convergent sequence of solutions satisfies the entropy inequality (4.8): this is the result stated in Lemma 4.3.6 below.

4.3.3 Passing to the limit in the scheme

The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem. In order to prove this theorem, we need some definitions of interpolates of regular test functions on the primal and dual meshes.

Definition 4.3.4 (Interpolates on one-dimensional meshes)**.** *Let Ω be an open bounded interval of $\mathbb{R}$, let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$, and let $\mathcal{M}$ be a mesh over Ω. The interpolate $\varphi_{\mathcal{M}}$ of φ on the primal mesh $\mathcal{M}$ is defined by:*

$$\varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \varphi_K^{n+1} \, \mathcal{X}_K \, \mathcal{X}_{[t^n,t^{n+1})},$$

where, for $0 \leq n \leq N$ and $K \in \mathcal{M}$, we set $\varphi_K^n = \varphi(x_K, t^n)$, with x_K the mass center of K. The time discrete derivative of the discrete function $\varphi_{\mathcal{M}}$ is defined by:

$$\eth_t \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} \, \mathcal{X}_K \, \mathcal{X}_{[t^n,t^{n+1})},$$

and its space discrete derivative by:

$$\eth_x \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma} \, \mathcal{X}_{D_\sigma} \, \mathcal{X}_{[t^n,t^{n+1})}.$$

Let $\varphi_{\mathcal{E}}$ be an interpolate of φ on the dual mesh, defined by:

$$\varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} \varphi_\sigma^{n+1} \, \mathcal{X}_{D_\sigma} \, \mathcal{X}_{[t^n,t^{n+1})},$$

where, for $1 \leq n \leq N$ and $\sigma \in \mathcal{E}$, we set $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$, with x_σ the abscissa of the interface σ. We also define the time and space discrete derivatives of this discrete function by:

$$\eth_t \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t} \, \mathcal{X}_{D_\sigma} \, \mathcal{X}_{[t^n,t^{n+1})},$$

$$\eth_x \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \frac{\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}}{h_K} \, \mathcal{X}_K \, \mathcal{X}_{[t^n,t^{n+1})}.$$

Finally, we define $\eth_x \varphi_{\mathcal{M},\mathcal{E}}$ by:

$$\eth_x \varphi_{\mathcal{M},\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_\sigma^{n+1}}{h_K/2} \, \mathcal{X}_{D_{K,\sigma}} \, \mathcal{X}_{[t^n,t^{n+1})} + \frac{\varphi_{\sigma'}^{n+1} - \varphi_K^{n+1}}{h_K/2} \, \mathcal{X}_{D_{K,\sigma'}} \, \mathcal{X}_{[t^n,t^{n+1})}.$$

For the consistency result that we are seeking (Theorem 4.3.5 below), we have to assume that a sequence of discrete solutions $\big(\rho^{(m)}, p^{(m)}, u^{(m)}\big)_{m\in\mathbb{N}}$ satisfies $\rho^{(m)} > 0$ and $p^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (4.30) is satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)^3$, *i.e.*:

$$0 < (\rho^{(m)})_K^n \leq C, \; 0 < (p^{(m)})_K^n \leq C, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \; \forall m \in \mathbb{N}, \tag{4.36}$$

and

$$|(u^{(m)})^n_\sigma| \le C, \quad \forall \sigma \in \mathcal{E}^{(m)}, \text{ for } 0 \le n \le N^{(m)}, \ \forall m \in \mathbb{N}, \tag{4.37}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0 and u_0 belong to $\mathrm{L}^\infty(\Omega)$. We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \le C, \quad \forall m \in \mathbb{N}. \tag{4.38}$$

We are not able to prove the estimates (4.36)–(4.38) for the solutions of the scheme; however, such inequalities are satisfied by the "interpolates" (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* with a limited sequence of meshes and time steps).

Theorem 4.3.5 (Consistency of the one-dimensional explicit scheme, barotropic case)**.** *Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (4.36)–(4.38) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^3$, for $1 \le p < \infty$, to $(\bar\rho, \bar p, \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^3$.*

Then the limit $(\bar\rho, \bar p, \bar u)$ satisfies the system (4.2).

Proof. It is clear that, with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The proof of this theorem is thus obtained by passing to the limit in the scheme for the mass balance equation first, and then for the momentum balance equation.

Mass balance equation – Let $\varphi \in \mathrm{C}^\infty_c(\Omega \times [0,T))$. Let $m \in \mathbb{N}$, $\mathcal{M}^{(m)}$ and $\delta t^{(m)}$ be given. Dropping for short the superscript $^{(m)}$, let $\varphi_\mathcal{M}$ be the interpolate of φ on the primal mesh and let $\eth_t \varphi_\mathcal{M}$ and $\eth_x \varphi_\mathcal{M}$ be its time and space discrete derivatives in the sense of Definition 4.3.4. Thanks to the regularity of φ, these functions respectively converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \ge 1$ (including $r = +\infty$), to φ, $\partial_t \varphi$ and $\partial_r \varphi$ respectively. In addition, $\varphi_\mathcal{M}(\cdot, 0)$ (which, for $K \in \mathcal{M}$ and $x \in K$, is equal to $\varphi^1_K = \varphi(x, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \ge 1$. Since the support of φ is compact in $\Omega \times [0,T)$, for m large enough, the interpolate of φ vanishes at the boundary cells and at the last time step(s); hereafter, we systematically assume that we are in this case.

Let us multiply the first equation (4.19b) of the scheme by $\delta t\, \varphi_K^{n+1}$, and sum the result for $0 \leq n \leq N-1$ and $K \in \mathcal{M}$, to obtain $T_1^{(m)} + T_2^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} |V_K| (\rho_K^{n+1} - \rho_K^n)\, \varphi_K^{n+1}, \qquad T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (F_{\sigma'}^n - F_\sigma^n)\, \varphi_K^{n+1}.$$

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} |V_K|\, \rho_K^n\, \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{K\in\mathcal{M}} |V_K|\, \rho_K^0\, \varphi_K^1,$$

so that:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} \eth_t \varphi_{\mathcal{M}}\, r^\alpha \,\mathrm{d}r \,\mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\, \varphi_{\mathcal{M}}(x,0)\, r^\alpha \,\mathrm{d}r.$$

The boundedness of ρ_0 and the definition (4.19a) of the initial conditions for the scheme ensures that the sequence $((\rho^{(m)})^0)_{m\in\mathbb{N}}$ converges to ρ_0 in $\mathrm{L}^r(\Omega)$ for $r \geq 1$. Since, by assumption, the sequence of discrete solutions and of the interpolate time derivatives converge in $\mathrm{L}^r\big(\Omega \times [0,T)\big)$ for $r \geq 1$, we thus obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \partial_t \varphi\, r^\alpha \,\mathrm{d}r \,\mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\, r^\alpha \,\mathrm{d}r.$$

Using the expression of the mass flux F_σ^n and reordering the sums in $T_2^{(m)}$, we get

$$T_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} h_\sigma\, r_\sigma^\alpha\, \rho_\sigma^n\, u_\sigma^n\, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}.$$

Since $|V_\sigma| = (|V_K| + |V_L|)/2$ and ρ_σ^n is the upwind approximation of ρ^n at the face σ, remarking that $|V_\sigma| = h_\sigma\, r_{\bar\sigma}^\alpha$ where $r_{\bar\sigma} \in (r_K, r_L)$, we can rewrite $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_1^{(m)} +$

$\mathcal{R}_2^{(m)}$ with

$$\mathcal{T}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \left(\frac{|V_K|}{2}\rho_K^n + \frac{|V_L|}{2}\rho_L^n\right) u_\sigma^n \, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

$$\mathcal{R}_1^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} h_\sigma \left(r_{\bar\sigma}^\alpha - r_\sigma^\alpha\right) \rho_\sigma^n \, u_\sigma^n \, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}$$

$$\mathcal{R}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} (\rho_K^n - \rho_L^n) \, \left[\frac{|V_K|}{2}(u_\sigma^n)^- + \frac{|V_L|}{2}(u_\sigma^n)^+\right] \, \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma},$$

where, for $a \in \mathbb{R}$, $a^+ = \max(a,0)$ and $a^- = -\min(a,0)$ (so $a = a^+ - a^-$). We have, for the term $\mathcal{T}_2^{(m)}$:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} u^{(m)} \eth_r \varphi_{\mathcal{M}} \, r^\alpha \, \mathrm{d}r \, \mathrm{d}t$$

and therefore, we obtain at the limit:

$$\lim_{m\to+\infty} \mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \bar\rho \, \bar u \, \partial_r \varphi \, r^\alpha \, \mathrm{d}r \, \mathrm{d}t.$$

The remainder terms $\mathcal{R}_1^{(m)}$ and $\mathcal{R}_2^{(m)}$ are bounded as follows, with $C_\varphi^r = \|\partial_r \varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}$:

$$|\mathcal{R}_1^{(m)}| \leq C_\varphi^r \, T \, \alpha \, |\Omega|^\alpha \, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \, h^{(m)},$$

$$|\mathcal{R}_2^{(m)}| \leq C_\varphi^r \, \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}} |\rho_K^n - \rho_L^n| \, |u_\sigma^n| \, |V_\sigma|$$

$$\leq C_\varphi^r \, \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \, |\Omega|^\alpha \, h^{(m)},$$

and therefore tend to zero when m tends to $+\infty$, by the assumed stability of the solution.

Momentum balance equation – Let $\varphi_\mathcal{E}$, $\eth_t \varphi_\mathcal{E}$ and $\eth_x \varphi_\mathcal{E}$ be the interpolate of φ on the dual mesh and its discrete time and space derivatives, in the sense of Definition 4.3.4, which converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \geq 1$ (including $r = +\infty$), to φ, $\partial_t \varphi$ and $\partial_r \varphi$ respectively. Let us multiply Equation (4.19d) by $\delta t \, \varphi_\sigma^{n+1}$, and sum the result for

$0 \leq n \leq N-1$ and $\sigma \in \mathcal{E}_{\rm int}$. We obtain $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\rm int}} |V_\sigma| \, (\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) \, \varphi_\sigma^{n+1},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}} \Big[F_L^n \, u_L^n - F_K^n \, u_K^n \Big] \, \varphi_\sigma^{n+1},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}} r_\sigma^\alpha \, (p_L^{n+1} - p_K^{n+1}) \, \varphi_\sigma^{n+1}.$$

Reordering the sums, we get for $T_1^{(m)}$:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\rm int}} |V_\sigma| \, \rho_{D_\sigma}^n u_\sigma^n \, \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t} - \sum_{\sigma \in \mathcal{E}_{\rm int}} |V_\sigma| \, \rho_{D_\sigma}^0 \, u_\sigma^0 \, \varphi_\sigma^1.$$

Thanks to the definition of the quantity ρ_{D_σ} (namely the fact that $|V_\sigma| \, \rho_{D_\sigma}^n = (|V_K| \, \rho_K^n + |V_L| \, \rho_L^n)/2$), we have:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} \, u^{(m)} \, \eth_t \varphi_{\mathcal{E}} \, r^\alpha \, \mathrm{d}r \, \mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x) \; (u^{(m)})^0(x) \; \varphi_{\mathcal{E}}(x,0) \, r^\alpha \, \mathrm{d}r.$$

By the same arguments as for the mass balance equation, we therefore obtain:

$$\lim_{m \to +\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho \, \bar u \, \partial_t \varphi \, r^\alpha \, \mathrm{d}r \, \mathrm{d}t - \int_\Omega \rho_0(x) \, u_0(x) \, \varphi(x,0) \, r^\alpha \, \mathrm{d}r.$$

Let us now turn to $T_2^{(m)}$. Reordering the sums and using the definition of the mass fluxes at the dual faces, we get:

$$\begin{aligned} T_2^{(m)} &= -\sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma \sigma'}] \in \mathcal{M}} F_K^n \, u_K^n \, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}) \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma \sigma'}] \in \mathcal{M}} (r_\sigma^\alpha \rho_\sigma^n u_\sigma^n + r_{\sigma'}^\alpha \rho_{\sigma'}^n u_{\sigma'}^n) \, u_K^n \, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}). \end{aligned}$$

Using the relation

$$\int_0^T \int_\Omega \rho^{(m)}\, {u^{(m)}}^2\, \eth_x \varphi_{\mathcal{E}}\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t \\ = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} r_{\bar K}^\alpha \rho_K^n \left[(u_\sigma^n)^2 + (u_{\sigma'}^n)^2\right] (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

with $r_{\bar K}^\alpha = |V_K|/h_K$, we can rewrite the term $T_2^{(m)}$ as

$$T_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, {u^{(m)}}^2\, \eth_x \varphi_{\mathcal{E}}\, r^\alpha \,\mathrm{d}r\,\mathrm{d}t + \mathcal{R}_1^{(m)} + \mathcal{R}_2^{(m)},$$

where:

$$\mathcal{R}_1^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \Big[(r_\sigma^\alpha - r_{\bar K}^\alpha)\, \rho_\sigma^n u_\sigma^n + (r_{\sigma'}^\alpha - r_{\bar K}^\alpha)\, \rho_{\sigma'}^n u_{\sigma'}^n\Big]\, u_K^n\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\mathcal{R}_2^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} r_{\bar K}^\alpha \Big[u_\sigma^n\, (\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n) + u_{\sigma'}^n\, (\rho_{\sigma'}^n u_K^n - \rho_K^n u_{\sigma'}^n)\Big] \\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

The first residual is bounded by the following inequality

$$|\mathcal{R}_1^{(m)}| \le C_\varphi^r\, T\, \alpha\, |\Omega|^\alpha\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))}\, h^{(m)}.$$

We now turn to the second one. At first, applying the identity $2(ab - cd) = (a-c)(b+d) + (a+c)(b-d),\ \forall (a,b,c,d) \in \mathbb{R}^4$, to the term $\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n$ and using the fact that the quantities $\rho_\sigma^n - \rho_K^n$ and $u_\sigma^n - u_K^n$ are either zero or differences of the density at two neighbouring cells and the velocity at two neighbouring faces respectively, then performing in the same manner to $\rho_{\sigma'}^n u_K^n - \rho_K^n u_{\sigma'}^n$, we obtain

$$|\mathcal{R}_2^{(m)}| \le C_\varphi^r\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))} \left(\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\right) h^{(m)}.$$

Therefore, the remainder term $\mathcal{R}_1^{(m)} + \mathcal{R}_2^{(m)}$ tends to zero when m tends to $+\infty$ and:

$$\lim_{m \to +\infty} T_2^{(m)} = -\int_0^T \int_\Omega \bar{\rho}\, \bar{u}^2\, \partial_r \varphi\, r^\alpha \,\mathrm{d}r \,\mathrm{d}t.$$

Let us finally study $T_3^{(m)}$. Reordering the sums, we obtain $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ with:

$$\mathcal{T}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, r_\sigma^\alpha\, \varphi_\sigma^{n+1},$$

$$\mathcal{R}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, r_\sigma^\alpha\, (\varphi_\sigma^n - \varphi_\sigma^{n+1}) - \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\rm int}} (p_L^0 - p_K^0)\, r_\sigma^\alpha\, \varphi_\sigma^1.$$

The bounds for remainder terms read

$$|\mathcal{R}_3^{(m)}| \leq \Big(\|\varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, C_p^0 + \|\partial_t \varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|p^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \Big)\, |\Omega|^\alpha\, \delta t^{(m)},$$

where C_p^0 is the bound of initial pressure all over computational domain. Therefore, the residual of $\mathcal{T}_3^{(m)}$ tends to zero when m tends to $+\infty$ and, since

$$\mathcal{T}_3^{(m)} = -\int_0^T \int_\Omega p^{(m)}\, \eth_r\big(r^\alpha \varphi_{\mathcal{E}}^{(m)}\big) \,\mathrm{d}r \,\mathrm{d}t,$$

we obtain that:

$$\lim_{m \to +\infty} T_3^{(m)} = -\int_0^T \int_\Omega \bar{p}\, \partial_r (r^\alpha \varphi) \,\mathrm{d}r \,\mathrm{d}t.$$

Conclusion – Gathering the limits of all the terms of the mass and momentum balance equations concludes the proof. ☐

We now turn to the entropy condition (4.8). To this purpose, we need to introduce the following additional condition for a sequence of discretizations:

$$\lim_{m \to +\infty} \frac{\delta t^{(m)}}{\min_{K \in \mathcal{M}^{(m)}} h_K} = 0. \tag{4.39}$$

Note that this condition is slightly more restrictive that a standard CFL condition. It allows to bound the remainder term in the discrete elastic potential balance as stated in the following lemma.

Lemma 4.3.6. *Let Ω be an open bounded interval of $\mathbb{R}$. Let $(\mathcal{M}^{(m)},\, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that the time step $\delta t^{(m)}$ tends to zero as $m \to \infty$, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (4.36)–(4.37)*. In addition, we assume that $(\rho^{(m)})_{m\in\mathbb{N}}$ satisfies the following uniform BV estimate:*

$$\|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}, \tag{4.40}$$

and, for $\gamma < 2$ only, is uniformly bounded by below, i.e. *that there exists $c > 0$ such that:*

$$c \leq (\rho^{(m)})^n_K, \quad \forall K \in \mathcal{M}^{(m)},\ \textit{for}\ 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{4.41}$$

Let us suppose that the CFL condition (4.39) *hold. Let $\mathcal{R}^{(m)}$ be defined by:*

$$\mathcal{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} (R_K^{n+1})^-,$$

with R_K^{n+1} given by (4.35)*. Then:*

$$\lim_{m\to+\infty} \mathcal{R}^{(m)} = 0.$$

Proof. For $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$, with $\sigma = \overrightarrow{M|K}$ and $\sigma' = \overrightarrow{K|L}$, we write $R_K^{n+1} = (T_1)_K^{n+1} + (T_2)_K^{n+1} + (T_3)_K^{n+1}$, with:

$$\begin{aligned}
(T_1)_K^{n+1} &= \frac{1}{2}\frac{|V_K|}{\delta t}\, \mathcal{H}''(\overline{\rho}^n_{K,1})\, (\rho_K^{n+1} - \rho_K^n)^2, \\
(T_2)_K^{n+1} &= \frac{1}{2}\Big[r^\alpha_{\sigma'}\, (u^n_{\sigma'})^-\, \mathcal{H}''(\overline{\rho}^n_{\sigma'})\, (\rho^n_K - \rho^n_L)^2 + r^\alpha_\sigma\, (u^n_\sigma)^+\, \mathcal{H}''(\overline{\rho}^n_\sigma)\, (\rho^n_K - \rho^n_M)^2\Big], \\
(T_3)_K^{n+1} &= \Big[r^\alpha_{\sigma'}\, \rho^n_{\sigma'}\, u^n_{\sigma'} - r^\alpha_\sigma\, \rho^n_\sigma\, u^n_\sigma\Big]\, \mathcal{H}''(\overline{\rho}^n_{K,2})\, (\rho_K^{n+1} - \rho^n_K),
\end{aligned}$$

where $\overline{\rho}^n_{K,1},\ \overline{\rho}^n_{K,2} \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, $\overline{\rho}^n_{\sigma'} \in [\![\rho^n_K, \rho^n_L]\!]$ and $\overline{\rho}^n_\sigma \in [\![\rho^n_K, \rho^n_M]\!]$. The first two terms are non-negative, and thus $(R_K^{n+1})^- \leq |(T_3)_K^{n+1}|$. Using the identity $2(ab - cd) = (a-c)(b+d) + (a+c)(b-d)$ and $(a-b)(a^\alpha + b^\alpha) = a^{\alpha+1} - b^{\alpha+1} + ab(b^{\alpha-1} - a^{\alpha-1})$,

$\forall (a,b,c,d) \in \mathbb{R}^4$ gives $(T_3)_K^{n+1} = (R_1)_K^{n+1} + (R_2)_K^{n+1} + (R_3)_K^{n+1}$ where

$$(R_1)_K^{n+1} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \frac{1}{2}(r_{\sigma'}^{\alpha} - r_{\sigma}^{\alpha})\,(\rho_{\sigma'}^n\, u_{\sigma'}^n + \rho_{\sigma}^n\, u_{\sigma}^n)\,(\rho_K^{n+1} - \rho_K^n)\,\mathcal{H}''(\overline{\rho}_{K,2}^n),$$

$$(R_2)_K^{n+1} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \frac{1}{2}(r_{\sigma'}^{\alpha+1} - r_{\sigma}^{\alpha+1})\,(\rho_{\sigma'}^n\, u_{\sigma'}^n - \rho_{\sigma}^n\, u_{\sigma}^n)\,(\rho_K^{n+1} - \rho_K^n)\,\mathcal{H}''(\overline{\rho}_{K,2}^n)\,\frac{1}{h_K},$$

$$(R_3)_K^{n+1} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \frac{1}{2} r_{\sigma'}^{\alpha+1}\, r_{\sigma}^{\alpha+1}\,(r_{\sigma}^{\alpha-1} - r_{\sigma'}^{\alpha-1})\,(\rho_{\sigma'}^n\, u_{\sigma'}^n - \rho_{\sigma}^n\, u_{\sigma}^n)\,(\rho_K^{n+1} - \rho_K^n)$$
$$\mathcal{H}''(\overline{\rho}_{K,2}^n)\,\frac{1}{h_K}.$$

We obtain from the expression (4.13) for the volumes of primal cells:

$$\frac{1}{2}(r_{\sigma'}^{\alpha} - r_{\sigma}^{\alpha}) \le |V_K|,$$
$$r_{\sigma'}^{\alpha+1}\, r_{\sigma}^{\alpha+1}\,(r_{\sigma'}^{\alpha-1} - r_{\sigma}^{\alpha-1}) \le |V_K|.$$

Since both ρ, u and, for $\gamma < 2$, $1/\rho$ are supposed to be bounded, there exists $C > 0$ such that:

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} |(T_3)_K^{n+1}| \le C \frac{\delta t^{(m)}}{\min_{K\in\mathcal{M}} h_K}\, \|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}},$$

which yields the conclusion by the assumption (4.39). $\square$

Then we are now in position to state the following consistency result.

Theorem 4.3.7 (Entropy consistency, barotropic case)**.** *Let the assumptions of Theorem* 4.3.5 *hold. Let us suppose in addition that the considered sequence of discretization satisfies* (4.39)*, and that* $(\rho^{(m)})_{m\in\mathbb{N}}$ *satisfies the BV estimate* (4.40) *and, for* $\gamma < 2$*, the uniform control* (4.41) *of* $1/\rho^{(m)}$ *. Then the limit* $(\bar\rho,\, \bar p,\, \bar u)$ *satisfies the entropy condition* (4.8)*.*

Proof. Let $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$, $\varphi \ge 0$. With the same notations for the interpolate of φ as in the preceding proof, we multiply the kinetic balance equation (4.31)-(4.32) by φ_σ^{n+1}, and the elastic potential balance (4.34)-(4.35) by φ_K^{n+1}, sum over the edges and cells respectively and over the time steps, to obtain the discrete version of (4.8):

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + \tilde T_1^{(m)} + \tilde T_2^{(m)} + \tilde T_3^{(m)} = -R^{(m)} - \tilde R^{(m)}$$

where:

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \frac{|V_K|}{\delta t} \big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\big]\ \varphi_K^{n+1},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \big[r_{\sigma'}^\alpha\, \mathcal{H}(\rho_{\sigma'}^n)\, u_{\sigma'}^n - r_\sigma^\alpha\, \mathcal{H}(\rho_\sigma^n)\, u_\sigma^n\big]\varphi_K^{n+1},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \big[p_K^n(r_{\sigma'}^\alpha\, u_{\sigma'}^n - r_\sigma^\alpha\, u_\sigma^n)\big]\ \varphi_K^{n+1},$$

$$\tilde{T}_1^{(m)} = \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm int}} \frac{|V_\sigma|}{\delta t} \big[\rho_{D_\sigma}^{n+1}(u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n(u_\sigma^n)^2\big]\ \varphi_\sigma^{n+1},$$

$$\tilde{T}_2^{(m)} = \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \big[F_L^n\,(u_L^n)^2 - F_K^n\,(u_K^n)^2\big]\ \varphi_\sigma^{n+1}, \qquad \text{with } K=[\overrightarrow{\sigma'\sigma}],\ L=[\overrightarrow{\sigma\sigma''}],$$

$$\tilde{T}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^{n+1} - p_K^{n+1})\, r_\sigma^\alpha\, u_\sigma^{n+1}\, \varphi_\sigma^{n+1},$$

$$R^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} R_K^{n+1}\ \varphi_K^{n+1}, \qquad \tilde{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm int}} R_\sigma^{n+1}\ \varphi_\sigma^{n+1},$$

and the quantities R_K^{n+1} and R_σ^{n+1} are given by Equation (4.35) and (4.32) respectively. By the same arguments as the proof of theorems 2.4.4 and 4.4.2, we obtain desired results.

□

4.4 The Euler equations

4.4.1 The scheme

The derivation of the explicit-in-time scheme for the Euler equations is obtained in the same manner to the barotropic Euler equations (Section 4.3). The fully discrete form of the scheme reads, for $0 \leq n \leq N-1$:

$$\begin{aligned}
&\forall K \in \mathcal{M}, \quad \rho_K^0 = \frac{1}{|V_K|} \int_K \rho_0(x)\, r^\alpha \,\mathrm{d}r, \qquad e_K^0 = \frac{1}{|V_K|} \int_K e_0(x)\, r^\alpha \,\mathrm{d}r, \\
&\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad u_\sigma^0 = \frac{1}{|V_\sigma|} \int_{D_\sigma} u_0(x)\, r^\alpha \,\mathrm{d}r,
\end{aligned} \tag{4.42a}$$

$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$,

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{4.42b}$$

$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$,

$$\frac{|V_K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^n e_{\sigma'}^n - F_\sigma^n e_\sigma^n + p_K^n (r_{\sigma'}^\alpha u_{\sigma'}^n - r_\sigma^\alpha u_\sigma^n) = S_K^n, \tag{4.42c}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1)\, \rho_K^{n+1}\, e_K^{n+1}, \tag{4.42d}$$

$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}$,

$$\frac{|V_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + r_\sigma^\alpha\,(p_L^{n+1} - p_K^{n+1}) = 0. \tag{4.42e}$$

The Equation (4.42b) and Equation (4.42e) are introduced in Section 4.3.1. Therefore, we describe only terms associated to the internal energy. The Equation (4.42c) is an approximation of the internal energy balance over the primal cell K. The positivity of the convection operator is ensured thanks to the upwinding choice for e_σ^n:

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}, \qquad e_\sigma^n = \left| \begin{array}{ll} e_K^n & \text{if } F_\sigma^n \geq 0, \\ e_L^n & \text{otherwise.} \end{array} \right.$$

The last term on the left-hand side is a natural approximation of the velocity derivative on primal cells which is given by (4.27). The right-hand side, S_K^n, is derived by using consistency arguments in the next section. Finally, the initial approximations for e is given by the average of the initial conditions e_0 on the primal cells.

4.4.2 Corrective source terms

The next step is now to define corrective terms in the internal energy balance, with the aim to recover a consistent discretization of the total energy balance. The first idea to do this could be just to sum the (discrete) kinetic energy balance with the internal energy balance: it is indeed possible for a collocated discretization. But here, we face the fact that the kinetic energy balance is associated to the dual mesh, while the internal energy balance is discretized on the primal one. The way to circumvent this difficulty is to remark that we do not really need a discrete total energy balance; in fact, we only need to recover (a weak form of) this equation when the mesh and time steps tend to zero. To this purpose, we choose the quantities (S_K^n) in such a way as to somewhat compensate the terms (R_σ^{n+1}) given by (4.32):

$$\begin{aligned}\forall K \in \mathcal{M},\ K = [\overrightarrow{\sigma\sigma'}], \qquad S_K^n = \frac{|V_K|}{4\,\delta t}\,\rho_K^n\,\big[(u_\sigma^n - u_\sigma^{n-1})^2 + (u_{\sigma'}^n - u_{\sigma'}^{n-1})^2\big] \\ + \frac{|F_K^{n-1}|}{2}\,(u_\sigma^{n-1} - u_{\sigma'}^{n-1})^2 + F_K^{n-1}\,(u_{\sigma'}^{n-1} - u_\sigma^{n-1})\,(u_K^n - u_K^{n-1}),\end{aligned} \tag{4.43}$$

where $u_K^n - u_K^{n-1}$ is a downwind choice with respect to F_K^{n-1}:

$$\forall K = \overrightarrow{\sigma|\sigma'} \in \mathcal{M}, \qquad u_K^n - u_K^{n-1} = \left| \begin{array}{ll} u_{\sigma'}^n - u_{\sigma'}^{n-1} & \text{if } F_K^{n-1} \geq 0, \\[2ex] u_\sigma^n - u_\sigma^{n-1} & \text{otherwise.} \end{array} \right.$$

The expression of the $(S_K^n)_{K\in\mathcal{M}}$ is justified by the passage to the limit in the scheme performed in the next section. Indeed, the first part of S_K^n, thanks to the expression (4.22) of the density at the face $\rho_{D_\sigma}^{n+1}$, results from a dispatching of the first part of the residual over the two adjacent cells:

$$\frac{1}{2}\,\frac{|V_\sigma|}{\delta t}\,\rho_{D_\sigma}^n\,\left(u_\sigma^n - u_\sigma^{n-1}\right)^2 = \underbrace{\frac{1}{2}\,\frac{|V_{K,\sigma}|}{\delta t}\,\rho_K^n\,\left(u_\sigma^n - u_\sigma^{n-1}\right)^2}_{\text{affected to K}} + \underbrace{\frac{1}{2}\,\frac{|V_{L,\sigma}|}{\delta t}\,\rho_L^n\,\left(u_\sigma^n - u_\sigma^{n-1}\right)^2}_{\text{affected to L}}.$$

The same argument holds for the terms associated to the dual faces. Therefore, the scheme conserves the integral of the total energy over the computational domain.

The definition (4.43) of $(S_K^n)_{K\in\mathcal{M}}$ allows to prove that, under a CFL condition, the scheme preserves the positivity of e.

Lemma 4.4.1. *Let us suppose that, for* $1 \leq n \leq N$ *and for all* $K = \sigma|\sigma' \in \mathcal{M}$*, we have:*

$$\delta t \leq \frac{|V_K|}{\gamma\big[r^{\alpha}_{\sigma'}\,(u^n_{\sigma'})^+ + r^{\alpha}_{\sigma}\,(u^n_{\sigma})^-\big]} \quad \textit{and} \quad \delta t \leq \frac{|V_K|\,\rho^n_K}{\big|F^{n-1}_{\sigma} + F^{n-1}_{\sigma'}\big|}. \tag{4.44}$$

Then the internal energy $(e^n)_{0 \leq n \leq N}$ *given by the scheme* (4.42) *is positive.*

Proof. We refer to Lemma 3.3.2 in Chapter 3 for the proof. ☐

4.4.3 Passing to the limit in the scheme

For the consistency result that we are seeking (Theorem 4.4.2 below), we have to assume that a sequence of discrete solutions $\big(\rho^{(m)}, p^{(m)}, u^{(m)}\big)_{m \in \mathbb{N}}$ satisfies $\rho^{(m)} > 0$, $p^{(m)} > 0$ and $e^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (4.30) is satisfied), and is uniformly bounded in $\mathrm{L}^{\infty}((0,T) \times \Omega)^3$, *i.e.*:

$$\begin{aligned} &\forall K \in \mathcal{M}^{(m)},\ \text{for } 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \\ &\qquad 0 < (\rho^{(m)})^n_K \leq C, \qquad 0 < (p^{(m)})^n_K \leq C, \qquad 0 < (e^{(m)})^n_K \leq C, \end{aligned} \tag{4.45}$$

and

$$|(u^{(m)})^n_{\sigma}| \leq C, \quad \forall \sigma \in \mathcal{E}^{(m)},\ \text{for } 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{4.46}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0, e_0 and u_0 belong to $\mathrm{L}^{\infty}(\Omega)$. We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|p^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|e^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}. \tag{4.47}$$

and:

$$\|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}. \tag{4.48}$$

We are not able to prove the estimates (4.45)–(4.48) for the solutions of the scheme; however, such inequalities are satisfied by the “interpolates” (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* with a limited sequence of meshes and time steps).

Theorem 4.4.2 (Consistency of the one-dimensional explicit scheme, Euler case)**.**
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$, $p_0 \in \mathrm{BV}(\Omega)$, $e_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to \infty$, and let $(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (4.45)–(4.48) *and converges in $\mathrm{L}^p(\Omega \times (0,T))^4$, for $1 \le p < \infty$, to $(\bar\rho, \bar p, \bar e, \bar u) \in \mathrm{L}^\infty(\Omega \times (0,T))^4$.*

Then the limit $(\bar\rho, \bar p, \bar e, \bar u)$ satisfies the system (4.12)*.*

Proof. It is clear that with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The fact that the limit satisfies the weak mass balance equation (4.12a) and the weak momentum balance equation (4.12b) is proven in Theorem 4.3.5. The proof of this theorem is thus obtained by passing to the limit in the scheme, in the internal and the kinetic energy balance equations.

Let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$. Let $m \in \mathbb{N}$, $\mathcal{M}^{(m)}$ and $\delta t^{(m)}$ be given. Dropping for short the superscript $^{(m)}$, let $\varphi_\mathcal{M}$ be the interpolate of φ on the primal mesh and let $\eth_t \varphi_\mathcal{M}$ and $\eth_x \varphi_\mathcal{M}$ be its time and space discrete derivatives in the sense of Definition 4.3.4. Thanks to the regularity of φ, these functions respectively converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \ge 1$ (including $r = +\infty$), to φ, $\partial_t \varphi$ and $\partial_x \varphi$ respectively. In addition, $\varphi_\mathcal{M}(\cdot, 0)$ (which, for $K \in \mathcal{M}$ and $x \in K$, is equal to $\varphi_K^1 = \varphi(x_K, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \ge 1$.

We also define $\varphi_\mathcal{E}$, $\eth_t \varphi_\mathcal{E}$ and $\eth_x \varphi_\mathcal{E}$, as, respectively, the interpolate of φ on the dual mesh and its discrete time and space derivatives, still in the sense of Definition 4.3.4; once again thanks to the regularity of φ, these functions converge in $\mathrm{L}^r(\Omega \times (0,T))$, for $r \ge 1$, to φ, $\partial_t \varphi$ and $\partial_x \varphi$ respectively. As for the interpolate on the primal mesh, $\varphi_\mathcal{E}(\cdot, 0)$ (which, for $\sigma \in \mathcal{E}$ and $x \in D_\sigma$, is equal to $\varphi_\sigma^1 = \varphi(x_\sigma, \delta t)$) converges to $\varphi(\cdot, 0)$ in $\mathrm{L}^r(\Omega)$ for $r \ge 1$.

Since the support of φ is compact in $\Omega \times [0,T)$, for m large enough, the interpolates of φ vanish on the boundary cells and at the last time step(s); hereafter, we systematically assume that we are in this case.

On one hand, let us multiply Equation (4.42c) by $\delta t\, \varphi_K^{n+1}$, and sum the result for $0 \le n \le N-1$ and $K \in \mathcal{M}$. On the second hand, let us multiply the discrete kinetic energy balance (4.31) by $\delta t\, \varphi_\sigma^{n+1}$, and sum the result over for $0 \le n \le N-1$ and

$\sigma \in \mathcal{E}_{\text{int}}$. Finally, adding the two obtained relations, we get:

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + \tilde{T}_1^{(m)} + \tilde{T}_2^{(m)} + \tilde{T}_3^{(m)} = S^{(m)} - \tilde{R}^{(m)}$$

where:

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} \frac{|V_K|}{\delta t} \left[\rho_K^{n+1}\, e_K^{n+1} - \rho_K^n\, e_K^n\right] \varphi_K^{n+1},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[r_{\sigma'}^\alpha\, \rho_{\sigma'}^n\, e_{\sigma'}^n\, u_{\sigma'}^n - r_\sigma^\alpha\, \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\right] \varphi_K^{n+1},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} p_K^n \left(r_{\sigma'}^\alpha\, u_{\sigma'}^n - r_\sigma^\alpha\, u_\sigma^n\right) \varphi_K^{n+1},$$

$$\tilde{T}_1^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|V_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n (u_\sigma^n)^2\right] \varphi_\sigma^{n+1},$$

$$\tilde{T}_2^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} \left[F_L^n\, (u_L^n)^2 - F_K^n\, (u_K^n)^2\right] \varphi_\sigma^{n+1}, \qquad \text{with } K = [\overrightarrow{\sigma'\sigma}],\ L = [\overrightarrow{\sigma\sigma''}],$$

$$\tilde{T}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} (p_L^{n+1} - p_K^{n+1})\, r_\sigma^\alpha\, u_\sigma^{n+1}\, \varphi_\sigma^{n+1},$$

$$S^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^n\, \varphi_K^{n+1}, \qquad \tilde{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} R_\sigma^{n+1}\, \varphi_\sigma^{n+1},$$

and the quantities S_K^n and R_σ^{n+1} are given by Equation (4.43) and (4.32) respectively.

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |V_K|\, \rho_K^n\, e_K^n\, \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{K \in \mathcal{M}} |V_K|\, \rho_K^0\, e_K^0\, \varphi_K^1,$$

so that:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, e^{(m)} \eth_t \varphi_{\mathcal{M}}\, r^\alpha \,\mathrm{d}r \,\mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\, (e^{(m)})^0(x)\, \varphi_{\mathcal{M}}(x,0)\, r^\alpha \,\mathrm{d}r.$$

The boundedness of ρ_0, e_0 and the definition (4.42a) of the initial conditions for the scheme ensures that the sequences $((\rho^{(m)})^0)_{m\in\mathbb{N}}$ and $((e^{(m)})^0)_{m\in\mathbb{N}}$ converge to ρ_0 and e_0 respectively in $\mathrm{L}^r(\Omega)$ for $r \geq 1$. Since, by assumption, the sequence of discrete solutions and of the interpolate time derivatives converge in $\mathrm{L}^r\big(\Omega \times [0,T)\big)$ for $r \geq 1$, we thus obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho\,\bar e\,\eth_t\varphi\, r^\alpha\,\mathrm{d}r\,\mathrm{d}t - \int_\Omega \rho_0(x)\; e_0(x)\;\varphi(x,0)\, r^\alpha\,\mathrm{d}r.$$

Reordering the sums in $T_2^{(m)}$, we get:

$$T_2^{(m)} = -\sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} h_\sigma\, r_\sigma^\alpha\, \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma}.$$

Using the relation

$$\int_0^T\int_\Omega \rho^{(m)}\, e^{(m)}\, u^{(m)}\, \eth_r\varphi_{\mathcal{M}}^{(m)}\, r^\alpha\,\mathrm{d}r\,\mathrm{d}t \\ = \sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \left(\frac{|V_K|}{2}\,\rho_K^n\, e_K^n + \frac{|V_L|}{2}\,\rho_L^n\, e_L^n\right) u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma},$$

we can rewrite $T_2^{(m)}$ as follow

$$T_2^{(m)} = -\int_0^T\int_\Omega \rho^{(m)}\, e^{(m)}\, u^{(m)}\, \eth_r\varphi_{\mathcal{M}}^{(m)}\, r^\alpha\,\mathrm{d}r\,\mathrm{d}t + \mathcal{R}_{2,1}^{(m)} + \mathcal{R}_{2,2}^{(m)},$$

where

$$\mathcal{R}_{2,1}^{(m)} = \sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \Big(|V_\sigma| - h_\sigma\, r_\sigma^\alpha\Big)\, \rho_\sigma^n\, e_\sigma^n\, u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma},$$

$$\mathcal{R}_{2,2}^{(m)} = \sum_{n=0}^{N-1}\delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \left[\frac{|V_K|}{2}\,\rho_K^n\, e_K^n + \frac{|V_L|}{2}\,\rho_L^n\, e_L^n - \left(\frac{|V_K|}{2}+\frac{|V_L|}{2}\right)\rho_\sigma^n\, e_\sigma^n\right] \\ u_\sigma^n\, \frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma}.$$

Using Taylor expansion for $|V_\sigma| - h_\sigma\, r_\sigma^\alpha$ and the upwind choice of $\rho_\sigma^n\, e_\sigma^n$ gives the bounds for remainder terms:

$$|\mathcal{R}_{2,1}^{(m)}| \leq T\,\alpha\,|\Omega|^{\alpha}\,C_{\varphi}^{r}\,\|\rho^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,\|e^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,\|u^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,h^{(m)},$$

$$|\mathcal{R}_{2,2}^{(m)}| \leq C_{\varphi}^{r}\sum_{n=0}^{N-1}\delta t\sum_{\sigma=K|L\in\mathcal{E}}\left(\frac{|V_K|}{2}+\frac{|V_L|}{2}\right)|u_\sigma^n|\,|\rho_L^n\,e_L^n-\rho_K^n\,e_K^n|.$$

Applying the identity $2\,(ab-cd)=(a-c)(b+d)+(a+c)(b-d)$, which holds for any $\{a,b,c,d\}\subset\mathbb{R}$, to the quantity $\rho_L^n\,e_L^n-\rho_K^n\,e_K^n$, we obtain:

$$|\mathcal{R}_{2,2}^{(m)}| \leq C_{\varphi}^{r}\,|\Omega|^{\alpha}\,h^{(m)}\,\|u^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,\Big[\|\rho^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,\|e^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \\ +\|e^{(m)}\|_{\mathrm{L}^{\infty}(\Omega\times(0,T))}\,\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\Big].$$

Thus, $|\mathcal{R}_{2,1}^{(m)}|+|\mathcal{R}_{2,2}^{(m)}|$ tends to zero when m tends to $+\infty$ and

$$\lim_{m\to+\infty}T_2^{(m)}=-\int_0^T\int_\Omega\bar\rho\,\bar e\,\bar u\,\partial_r\varphi\,r^{\alpha}\,\mathrm{d}r\,\mathrm{d}t.$$

For the term $\tilde T_1^{(m)}$, the definition (4.22) of ρ_{D_σ} yields:

$$\tilde T_1^{(m)}=-\sum_{n=0}^{N-1}\delta t\sum_{\sigma=K|L\in\mathcal{E}}\Big[|V_{K,\sigma}|\,\rho_K^n+|V_{L,\sigma}|\,\rho_L^n\Big]\,u_\sigma^n\,\frac{\varphi_K^{n+1}-\varphi_K^n}{\delta t} \\ -\sum_{\sigma=K|L\in\mathcal{E}}\Big[|V_{K,\sigma}|\,\rho_K^0+|V_{L,\sigma}|\,\rho_L^0\Big]\,u_\sigma^0\,\varphi_K^1,$$

so, by similar arguments as for the term $T_1^{(m)}$, we get:

$$\lim_{m\to+\infty}\tilde T_1^{(m)}=-\int_0^T\int_\Omega\frac{1}{2}\,\bar\rho\,\bar u^2\,\partial_t\varphi\,r^{\alpha}\,\mathrm{d}r\,\mathrm{d}t-\int_\Omega\frac{1}{2}\,\rho_0(x)\,u_0(x)^2\,\varphi(x,0)\,r^{\alpha}\,\mathrm{d}r.$$

Let us now to the term $\tilde T_2^{(m)}$. Reordering the sums, we get:

$$\tilde T_2^{(m)}=-\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}}F_K^n\,(u_K^n)^2\,(\varphi_{\sigma'}^{n+1}-\varphi_\sigma^{n+1}),$$

The upwind choice of u_K^n with respect to F_K^n allows to write $\tilde{T}_2^{(m)} = \tilde{T}_{2,1}^{(m)} + \tilde{T}_{2,2}^{(m)}$ with

$$\tilde{T}_{2,1}^{(m)} = -\frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} F_K^n \left[(u_\sigma^n)^2 + (u_{\sigma'}^n)^2\right](\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\tilde{T}_{2,2}^{(m)} = -\frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} |F_K^n| \left[(u_\sigma^n)^2 - (u_{\sigma'}^n)^2\right](\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

Thanks to the definition of the mass flux at dual edges, $\tilde{T}_{2,1}^{(m)}$ turns out

$$\tilde{T}_{2,1}^{(m)} = -\frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \frac{1}{2}\left(r_\sigma^\alpha\,\rho_\sigma^n u_\sigma^n + r_{\sigma'}^\alpha\,\rho_{\sigma'}^n u_{\sigma'}^n\right)\left[(u_\sigma^n)^2 + (u_{\sigma'}^n)^2\right](\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

Using the identity $2\,(a^3+b^3) = (a+b)(a-b)^2 + (a+b)(a^2+b^2)$, which holds for any a, $b \in \mathbb{R}$, to the quantity $(u_\sigma^n)^3 + (u_{\sigma'}^n)^3$, and the relation

$$\int_0^T \int_\Omega \frac{1}{2}\,\rho^{(m)}\,(u^{(m)})^3\,\eth_r\varphi_{\mathcal{E}}\,r^\alpha\,\mathrm{d}r\,\mathrm{d}t = \frac{1}{4}\sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} r_{\bar{K}}^\alpha\,\rho_K^n\left[(u_\sigma^n)^3 + (u_{\sigma'}^n)^3\right](\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

where $r_{\bar{K}}^\alpha = |V_K|/h_K$, gives

$$\tilde{T}_{2,1}^{(m)} = -\int_0^T \int_\Omega \frac{1}{2}\,\rho^{(m)}\,(u^{(m)})^3\,\eth_r\varphi_{\mathcal{E}}\,r^\alpha\,\mathrm{d}r\,\mathrm{d}t + \tilde{\mathcal{R}}_{2,1}^{(m)} + \tilde{\mathcal{R}}_{2,2}^{(m)} + \tilde{\mathcal{R}}_{2,3}^{(m)},$$

with

$$\tilde{\mathcal{R}}_{2,1}^{(m)} = -\frac{1}{8} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \Big[(r_\sigma^\alpha - r_{\bar K}^\alpha)\, \rho_\sigma^n\, u_\sigma^n + (r_{\sigma'}^\alpha - r_{\bar K}^\alpha)\, \rho_{\sigma'}^n\, u_{\sigma'}^n \Big]\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\tilde{\mathcal{R}}_{2,2}^{(m)} = \frac{1}{8} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} r_{\bar K}^\alpha\, \rho_K^n\, (u_\sigma^n + u_{\sigma'}^n)\, (u_\sigma^n - u_{\sigma'}^n)^2\, (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$

$$\tilde{\mathcal{R}}_{2,3}^{(m)} = -\frac{1}{8} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} r_{\bar K}^\alpha \left[(\rho_\sigma^n - \rho_K^n)\, u_\sigma^n + (\rho_{\sigma'}^n - \rho_K^n)\, u_{\sigma'}^n) \right] \left[(u_\sigma^n)^2 + (u_{\sigma'}^n)^2 \right]$$
$$(\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

In a similar way as preceding proofs, we obtain the bounds for remainder terms as follow

$$\begin{aligned}
|\tilde{\mathcal{R}}_{2,1}^{(m)}| &\leq T\, \alpha\, |\Omega|^\alpha\, C_\varphi^r\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}^3\, h^{(m)},\\
|\tilde{\mathcal{R}}_{2,2}^{(m)}| &\leq |\Omega|^\alpha\, C_\varphi^r\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}^2\, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\, h^{(m)},\\
|\tilde{\mathcal{R}}_{2,3}^{(m)}| &\leq |\Omega|^\alpha\, C_\varphi^r\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}^3\, \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\, h^{(m)},\\
|\tilde{\mathcal{T}}_{2,2}^{(m)}| &\leq |\Omega|^\alpha\, C_\varphi^r\, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}^2\, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\, h^{(m)}.
\end{aligned}$$

and hence:

$$\lim_{m\to+\infty} \tilde{\mathcal{T}}_2^{(m)} = -\int_0^T \int_\Omega \frac{1}{2}\, \bar\rho\, \bar u^3\, \partial_r\varphi\, r^\alpha\, \mathrm{d}r\, \mathrm{d}t.$$

We now turn to $T_3^{(m)}$ and $\tilde{T}_3^{(m)}$. By a change in the notation of the time exponents, using the fact that φ_σ vanishes at the last time step(s), we get:

$$\tilde{T}_3^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} (p_L^n - p_K^n)\, r_\sigma^\alpha\, u_\sigma^n\, \varphi_\sigma^n = \tilde{\mathcal{T}}_3^{(m)} + \tilde{\mathcal{R}}_3^{(m)},$$

with:

$$\tilde{\mathcal{T}}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, r_\sigma^\alpha\, u_\sigma^n\, \varphi_\sigma^{n+1}$$
$$= -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\, (r_{\sigma'}^\alpha\, u_{\sigma'}^n\, \varphi_{\sigma'}^{n+1} - r_\sigma^\alpha\, u_\sigma^n\, \varphi_\sigma^{n+1}),$$
$$\tilde{\mathcal{R}}_3^{(m)} = \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^0 - p_K^0)\, r_\sigma^\alpha\, u_\sigma^0\, \varphi_\sigma^1 + \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} (p_L^n - p_K^n)\, r_\sigma^\alpha\, u_\sigma^n\, (\varphi_\sigma^n - \varphi_\sigma^{n+1}).$$

We have, thanks to the regularity of φ:

$$|\tilde{\mathcal{R}}_3^{(m)}| \le C_\varphi\, |\Omega|^\alpha\, \delta t^{(m)} \Big[\|(u^{(m)})^0\|_{\mathrm{L}^\infty(\Omega)}\, \|(p^{(m)})^0\|_{\mathrm{BV}(\Omega)} + \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|p^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \Big].$$

Therefore, invoking the regularity of the initial conditions, this term tends to zero when m tends to $+\infty$. In the next step, we take the sum of $T_3^{(m)}$ and $\tilde{\mathcal{T}}_3^{(m)}$ and make appear the coefficient $r_{\bar K}^\alpha$ to obtain $T_3^{(m)} + \tilde{\mathcal{T}}_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ with:

$$\mathcal{T}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\, r_{\bar K}^\alpha \left[u_{\sigma'}^n\, (\varphi_K^{n+1} - \varphi_{\sigma'}^{n+1}) - u_\sigma^n\, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}) \right]$$
$$= -\int_0^T \int_\Omega p^{(m)}\, u^{(m)}\, \eth_x \varphi_{\mathcal{M},\mathcal{E}}\, r^\alpha \,\mathrm{d}r\, \mathrm{d}t,$$
$$\mathcal{R}_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n \Big[(r_{\sigma'}^\alpha - r_{\bar K}^\alpha)\, u_{\sigma'}^n\, (\varphi_K^{n+1} - \varphi_{\sigma'}^{n+1}) - (r_\sigma^\alpha - r_{\bar K}^\alpha)\, u_\sigma^n\, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}) \Big].$$

In the similar way as preceding proofs, we have the bound for the remainder term:

$$|\mathcal{R}_3^{(m)}| \le T\, \alpha\, |\Omega|^\alpha\, C_\varphi^r\, \|p^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\, h^{(m)}.$$

So, since $\eth_x \varphi_{\mathcal{M},\mathcal{E}}$ converges to $\partial_x \varphi$ in $\mathrm{L}^r(\Omega \times (0,T))$ for any $r \geq 1$, we get:

$$\lim_{m \to +\infty} T_3^{(m)} + \tilde{\mathcal{T}}_3^{(m)} = -\int_0^T \int_\Omega \bar{p}\, \bar{u}\, \partial_r \varphi\, r^\alpha \,\mathrm{d}r \,\mathrm{d}t.$$

Finally, it now remains to check that $\lim_{m \to +\infty} S^{(m)} - \tilde{R}^{(m)} = 0$. Let us write this quantity as $S^{(m)} - \tilde{R}^{(m)} = \mathcal{R}_1^{(m)} + \mathcal{R}_2^{(m)}$ where, using $S_K^0 = 0, \forall K \in \mathcal{M}$:

$$\mathcal{R}_1^{(m)} = \sum_{n=0}^{N-1} \delta t \big[\sum_{K \in \mathcal{M}} S_K^n \varphi_K^{n+1} - \sum_{\sigma \in \mathcal{E}} R_\sigma^n \varphi_\sigma^{n+1} \big], \qquad \mathcal{R}_2^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^n \, (\varphi_K^{n+1} - \varphi_K^n).$$

First, we prove that $\lim_{m \to +\infty} \mathcal{R}_1^{(m)} = 0$. Gathering and reordering sums, we obtain $\mathcal{R}_1^{(m)} = \mathcal{R}_{1,1}^{(m)} + \mathcal{R}_{1,2}^{(m)} + \mathcal{R}_{1,3}^{(m)}$ with

$$\begin{aligned}
\mathcal{R}_{1,1}^{(m)} &= \frac{1}{4} \sum_{n=0}^{N-1} \sum_{\sigma = K|L \in \mathcal{E}} \Big[|V_K|\, \rho_K^{n+1}\, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}) + |V_L|\, \rho_L^{n+1}\, (\varphi_L^{n+1} - \varphi_\sigma^{n+1}) \Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u_\sigma^{n+1} - u_\sigma^n)^2, \\
\mathcal{R}_{1,2}^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |F_K^n| \, (u_\sigma^n - u_{\sigma'}^n)^2 \, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}), \\
\mathcal{R}_{1,3}^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K = [\sigma' \to \sigma] \in \mathcal{M}} F_K^n \, (u_{\sigma'}^n - u_\sigma^n) \, (u_\sigma^{n+1} - u_\sigma^n) \, (\varphi_K^{n+1} - \varphi_\sigma^{n+1}).
\end{aligned}$$

We thus obtain:

$$|\mathcal{R}_{1,1}^{(m)}| \leq h^{(m)}\, C_\varphi \, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega \times (0,T))} \, \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega \times (0,T))} \, \|u^{(m)}\|_{\mathcal{T},t,\mathrm{BV}},$$

and $$|\mathcal{R}_{1,2}^{(m)}| + |\mathcal{R}_{1,3}^{(m)}| \leq 2\, h^{(m)}\, C_\varphi^r \, \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega \times (0,T))} \, \|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega \times (0,T))} \, \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}},$$

so all these terms tend to zero. The fact that $|\mathcal{R}_2^{(m)}|$ behaves as $\delta t^{(m)}$ may be proven by very similar arguments.

Conclusion – Gathering the limits of all the terms concludes the proof. $\square$

4.5 Numerical results

We assess in this section the behaviour of the scheme(s) on explosion and implosion test cases. To this purpose, we address the Riemann problem studied in [61, Chapter 17]. For the barotropic Euler equations, we choose $p = \rho^2$ for the equation of state, so the solved system turns out to be the so-called shallow water equations.

4.5.1 Shallow water equations

Let us consider the implosion on cylindrical coordinate system, when $\alpha = 1$. The initial data consisting in two constant states separated by a discontinuity are chosen to obtain circular shock wave travelling towards the center, a circular contact surface travelling in the same direction and a circular rarefaction travelling way from the origin. The computational domain is the square $\Omega = [0, 2] \times [0, 2]$. The initial conditions consist of the region inside of a circle of radius $R = 0.4$ centred at $(1, 1)$ and the region outside the circle:

$$\text{inside state: } \begin{bmatrix} \rho_{ins} = 1 \\ u_{ins} = 2 \end{bmatrix}; \qquad \text{outside state: } \begin{bmatrix} \rho_{out} = 2 \\ u_{out} = 2 \end{bmatrix}.$$

The density, velocity and pressure obtained at the final time $T = 0.01$ with $h = 1/800$ and $\delta t = h/10$ are shown of Figures 4.1, 4.2, and 4.3 respectively, where the two-dimensional solution along the radial line that is coincident with the x–axis.

4.5.2 Euler equations

For the full Euler equations, we refer to [61, Test case 17.3] for a spherical explosion test (corresponding to the case $\alpha = 2$), with two constant states given by:

$$\text{inside state: } \begin{bmatrix} \rho_{ins} = 1 \\ u_{ins} = 0 \\ p_{ins} = 1 \end{bmatrix}; \qquad \text{outside state: } \begin{bmatrix} \rho_{out} = 0.125 \\ u_{out} = 0 \\ p_{out} = 0.1 \end{bmatrix}.$$

These initial conditions give the inverse structure of waves in case of shallow water equations. In detail, we obtain a circular shock wave travelling away from the centre, a circular contact surface travelling in the same direction and a circular rarefaction travelling towards the origin $(1, 1, 1)$. The computational domain is the cube $\Omega = [0, 2] \times [0, 2] \times [0, 2]$.

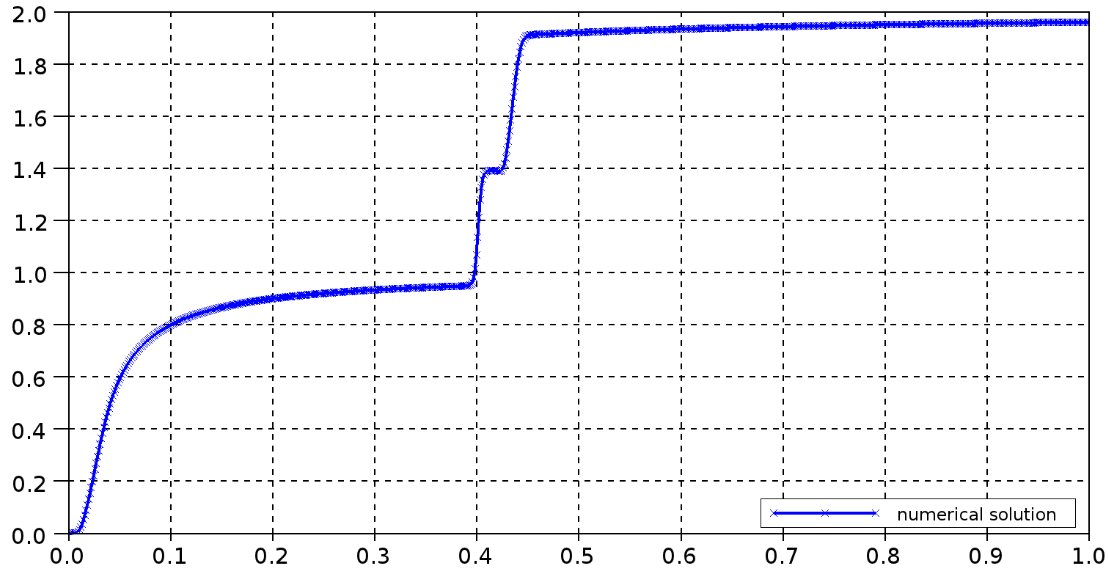


Figure 4.1: Cylindrical implosion – $h = 1/800,\ \delta t = h/10$ – Density at $T = 0.01$.

The three-dimensional solutions including density, velocity, pressure and internal energy obtained along the radial line that is coincident with the x–axis at the final time $T = 0.25$ with $h = 1/800$ and $\delta t = h/10$ are shown of Figures 4.4, 4.5, 4.6 and 4.7 respectively. We do not an exact solution for the three-dimensional Euler equations, however, the numerical solutions obtained by our scheme are compatible with the reference solutions in [61, Figure 17.7].

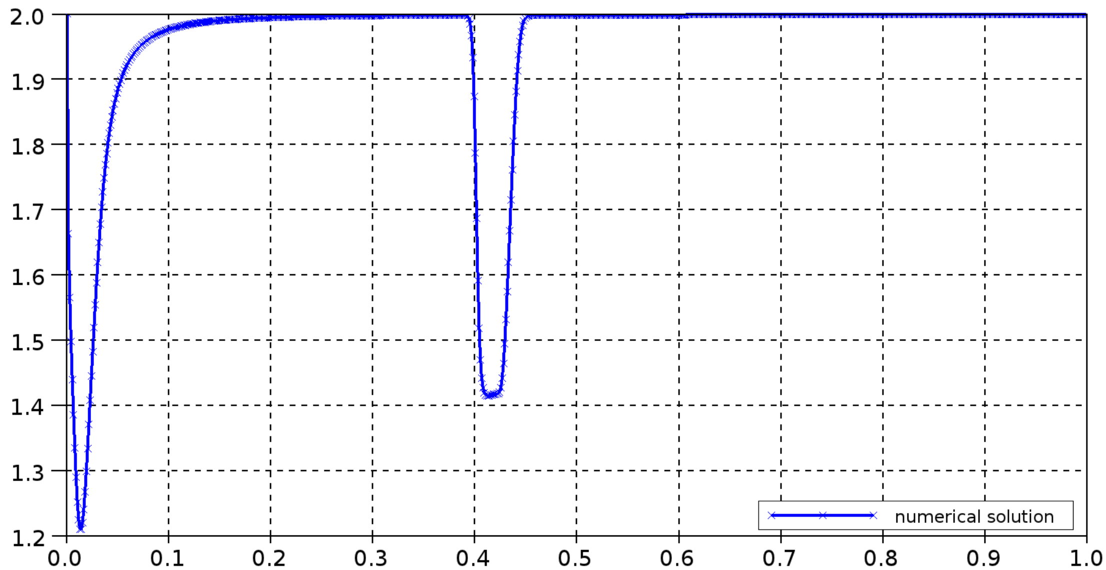


Figure 4.2: Cylindrical implosion – $h = 1/800,\ \delta t = h/10$ – Velocity at $T = 0.01$.

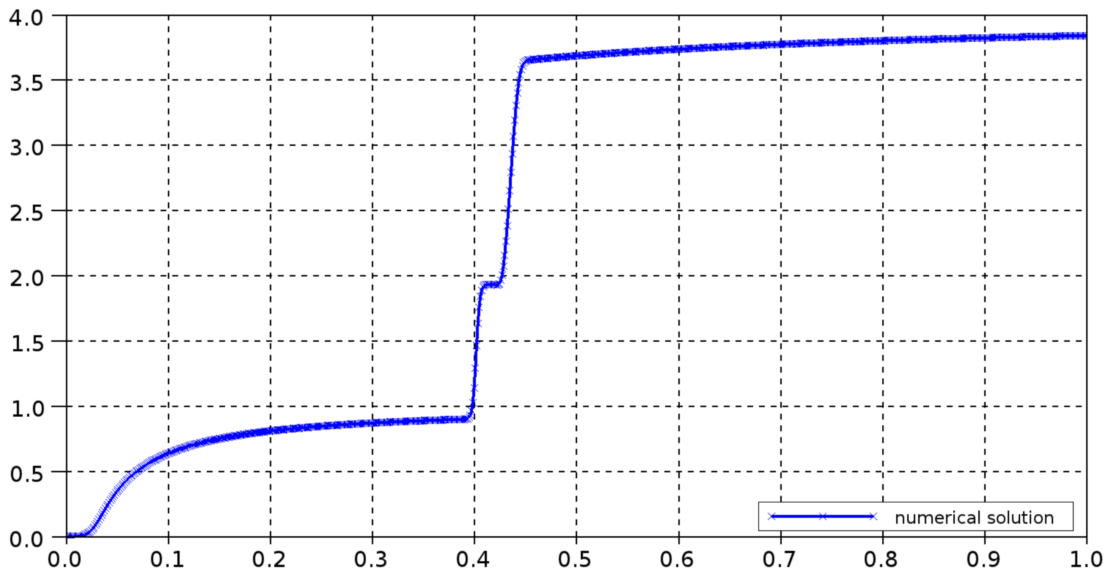


Figure 4.3: Cylindrical implosion – $h = 1/800,\ \delta t = h/10$ – Pressure at $T = 0.01$.

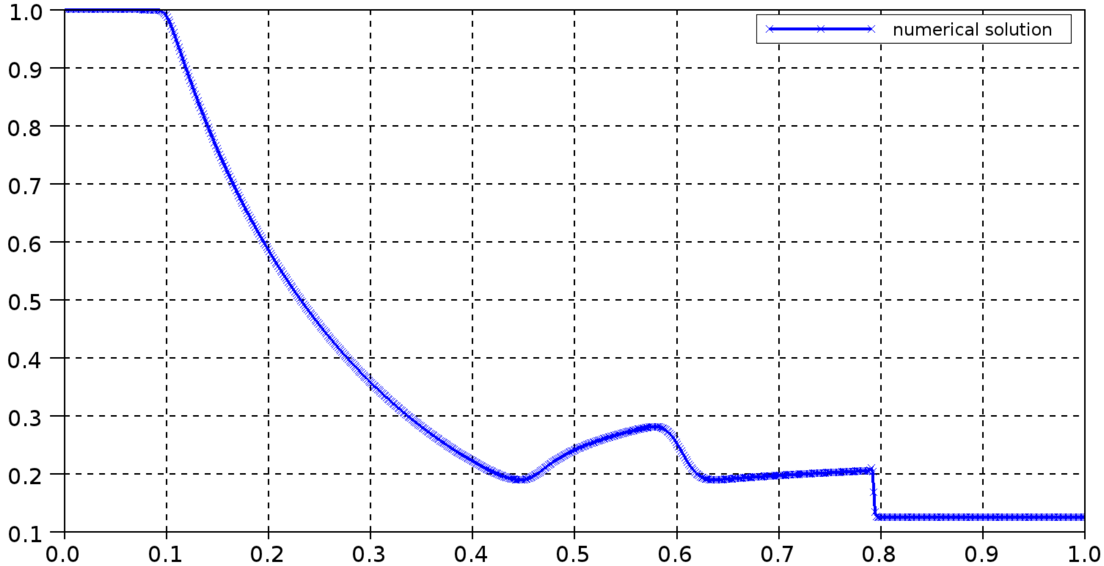


Figure 4.4: Spherical explosion – $h = 1/800,\ \delta t = h/10$ – Density at $T = 0.25$.

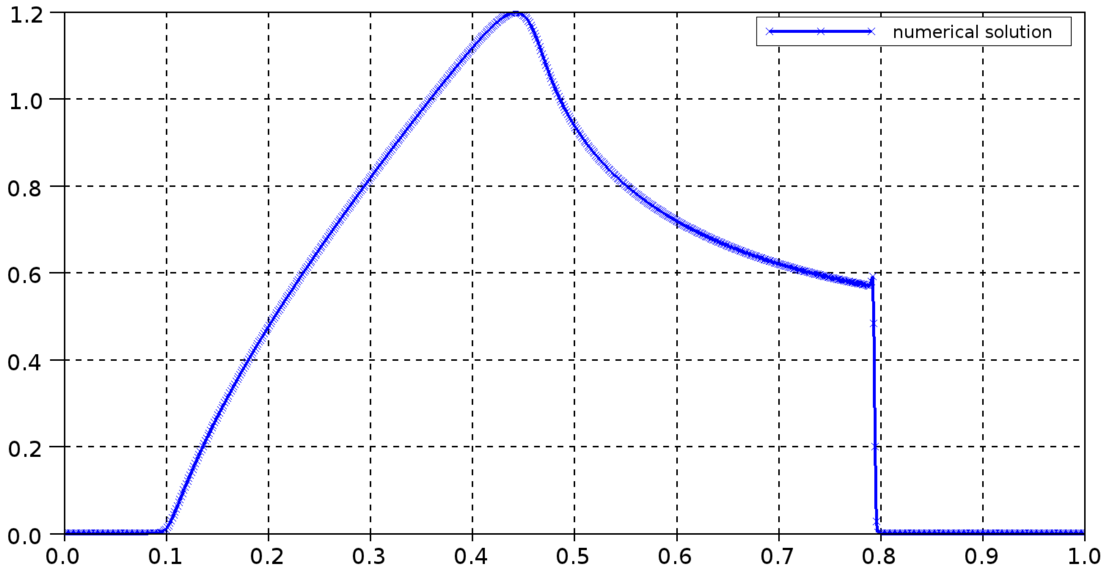


Figure 4.5: Spherical explosion – $h = 1/800,\ \delta t = h/10$ – Velocity at $T = 0.25$.

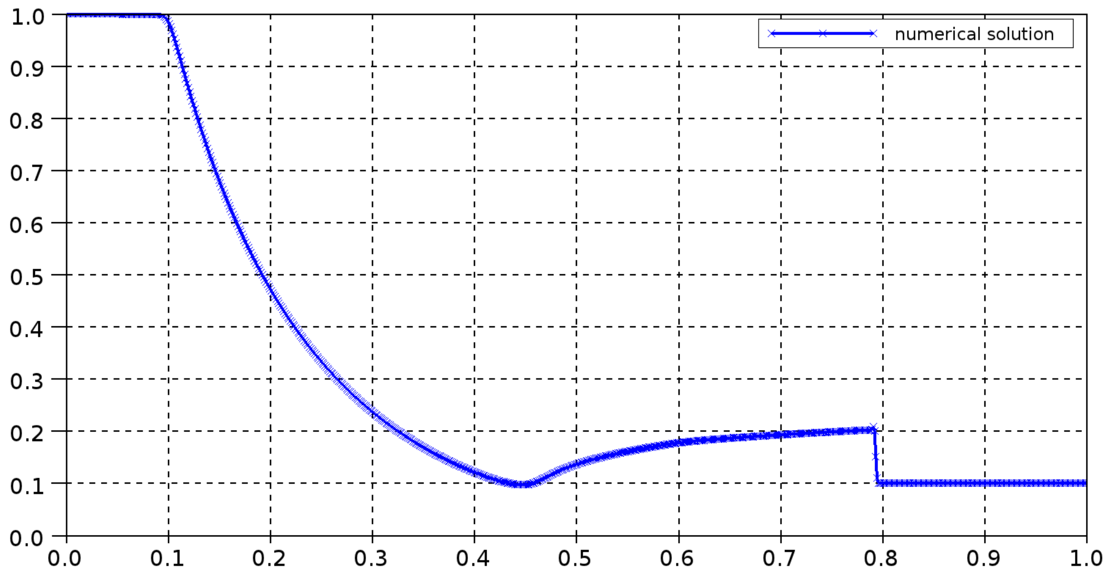


Figure 4.6: Spherical explosion – $h = 1/800,\ \delta t = h/10$ – Pressure at $T = 0.25$.

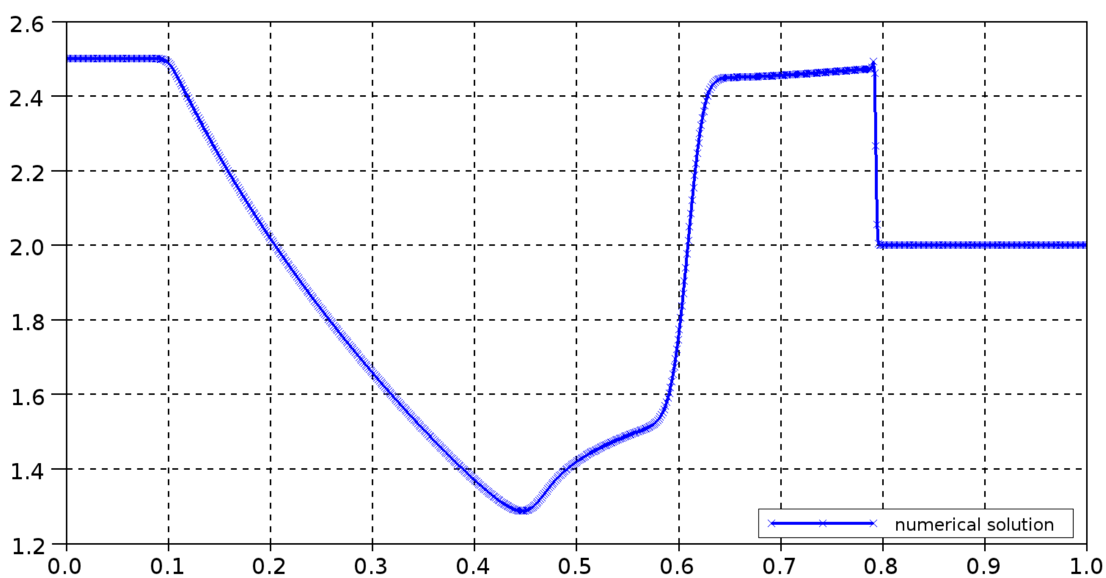


Figure 4.7: Spherical explosion – $h = 1/800,\ \delta t = h/10$ – Internal Energy at $T = 0.25$.

Appendix A

Exact solutions for the shallow water equations

A.1 Non-vacuum case

Let us recall the conservative form of the one–dimensional shallow water equations in case of ideal gas:

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{A.1a}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0, \tag{A.1b}$$

$$p = \rho^2, \tag{A.1c}$$

The sound speed a corresponding to the equation of state (A.1c) is given by:

$$a = \sqrt{p'(\rho)} = \sqrt{2\rho}. \tag{A.2}$$

Let us rewrite Equation (A.1a) and (A.1b) in differential form:

$$\mathbb{U}_t + \mathbb{F}(\mathbb{U})_x = 0, \tag{A.3}$$

where $\mathbb{U}$ and $\mathbb{F}(\mathbb{U})$ are the vectors of conserved variables and fluxes, given respectively by:

$$\mathbb{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \qquad \mathbb{F}(\mathbb{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix} = \begin{bmatrix} u_2 \\ u_2^2/u_1 + u_1^2 \end{bmatrix}. \tag{A.4}$$

The conservation laws (A.3)-(A.4) can be written in quasi-linear form:

$$\mathbb{U}_t + \mathbb{A}(\mathbb{U})\mathbb{U}_x = 0, \tag{A.5}$$

where the coefficient matrix $\mathbb{A}(\mathbb{U})$ is the Jacobian matrix:

$$\mathbb{A}(\mathbb{U}) = \begin{pmatrix} 0 & 1 \\ -\left(\frac{u_2}{u_1}\right)^2 + 2u_1 & 2\frac{u_2}{u_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + a^2 & 2u \end{pmatrix} \tag{A.6}$$

and the eigenvalues of the Jacobian matrix $\mathbb{A}(\mathbb{U})$ are

$$\lambda_1 = u - a, \qquad \lambda_2 = u + a. \tag{A.7}$$

The corresponding right eigenvectors are given by:

$$K^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \qquad K^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix}. \tag{A.8}$$

Hence, we have two waves associated with the two genuinely non-linear characteristic field $K^{(1)}$ and $K^{(2)}$:

$$\nabla\lambda_1 \cdot K^{(1)} = \begin{bmatrix} -\frac{u}{\rho} \\ \frac{1}{\rho} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ u - a \end{bmatrix} = -\frac{a}{\rho} \neq 0,$$
$$\nabla\lambda_2 \cdot K^{(2)} = \begin{bmatrix} -\frac{u}{\rho} \\ \frac{1}{\rho} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ u + a \end{bmatrix} = \frac{a}{\rho} \neq 0.$$

Roughly speaking, the two waves are either shock or rarefaction. The two waves separate the relevant domain of interest $x_L < x < x_R$, $t > 0$, with $x_L < 0$ and $x_R > 0$ three constant states. From left to right these are $\mathbb{U}_L$ (left data state), $\mathbb{U}_*$ (Star Region) and $\mathbb{U}_R$ (right data state). The complete solution of the shallow water equations is given by observing the structure of each wave. The Generalized Riemann Invariants across

λ_1-wave and λ_2-wave, are respectively

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u-a} \Leftrightarrow du + \frac{a}{\rho}d\rho = 0 \Leftrightarrow I_L(\rho,\rho u) := u + \int \frac{a}{\rho}d\rho = u + 2a = \text{const}, \quad \text{(A.9)}$$

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u+a} \Leftrightarrow du - \frac{a}{\rho}d\rho = 0 \Leftrightarrow I_R(\rho,\rho u) := u - \int \frac{a}{\rho}d\rho = u - 2a = \text{const}. \quad \text{(A.10)}$$

Let us consider the case of left shock wave. We denote $W_L = (\rho_L, u_L, p_L)$ and $W_* = (\rho_*, u_*, p_*)$ pre-shock and post-shock values, respectively. The entropy condition

$$\lambda_L(\mathbb{U}_L) > S_1 > \lambda_*(\mathbb{U}_*) \quad \text{(A.11)}$$

deduces that $S_1 < u_L$ where S_1 is the shock speed. We transform the problem to a new frame of reference moving with the shock so that in the new frame the shock speed is 0

$$\hat{u}_L = u_L - S_1 > 0, \qquad \hat{u}_* = u_* - S_1.$$

The Rankine-Hugoniot conditions give:

$$\rho_L \hat{u}_L = \rho_* \hat{u}_* =: Q_L \;, \quad \text{(A.12)}$$

$$\rho_L \hat{u}_L^2 + p_L = \rho_* \hat{u}_*^2 + p_* \;. \quad \text{(A.13)}$$

Using (A.12)-(A.13) and solving for Q_L yield:

$$-\frac{p_* - p_L}{\hat{u}_* - \hat{u}_L} = Q_L = -\frac{p_* - p_L}{u_* - u_L}. \quad \text{(A.14)}$$

Applying (A.12) and the equation of state (A.1c) on the first identity of (A.14), we obtain:

$$Q_L^2 = -\frac{(\rho_*^2 - \rho_L^2)}{\frac{1}{\rho_*} - \frac{1}{\rho_*}} = \rho_* \rho_L(\rho_* + \rho_L). \quad \text{(A.15)}$$

From the second identity of (A.14), we have the equation of the velocity in the Star Region:

$$u_* = u_L - \frac{p_* - p_L}{Q_L} = u_L - \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_L}\right)}(\rho_* - \rho_L). \quad \text{(A.16)}$$

On the other hand, the Rankine-Hugoniot in the original frame

$$S_1 = \frac{\rho_L u_L - \rho_* u_*}{\rho_L - \rho_*} \tag{A.17}$$

gives S_1 as a function of the density ρ_*:

$$S_1 = \frac{\rho_L u_L - \rho_* \left[u_L - \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_L}\right)}(\rho_* - \rho_L) \right]}{\rho_L - \rho_*} = u_L - \rho_* \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_L}\right)}. \tag{A.18}$$

Thus, the entropy condition (A.11) yields:

$$u_L - \sqrt{2\rho_L} > u_L - \rho_* \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_L}\right)},$$

or

$$\rho_* > \rho_L. \tag{A.19}$$

For the right shock wave, in the similar way of computations in the case of left shock, we obtain:

$$S_2 = u_R + \rho_* \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_R}\right)}\ , \tag{A.20}$$

$$u_* = u_R - \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_R}\right)}(\rho_R - \rho_*)\ , \tag{A.21}$$

$$\rho_* > \rho_R\ . \tag{A.22}$$

We now turn to the rarefaction wave. Using Equation (A.9) and (A.10) gives the Generalized Riemann Invariants for left and right states:

Left rarefaction: $$u_L + 2a_L = u_* + 2a_* \Leftrightarrow u_* = u_L + 2\sqrt{2}(\sqrt{\rho_L} - \sqrt{\rho_*}). \tag{A.23}$$

Right rarefaction: $$u_R - 2a_R = u_* - 2a_* \Leftrightarrow u_* = u_R - 2\sqrt{2}(\sqrt{\rho_R} - \sqrt{\rho_*}). \tag{A.24}$$

Let us summary the crucial results from researching four types of waves which are able to appear in our problem:

- Left shock:

$$u_* = u_L - \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_L}\right)(\rho_* - \rho_L)}, \tag{A.25}$$

$$\rho_* > \rho_L, \qquad u_* < u_L. \tag{A.26}$$

- Right shock:

$$u_* = u_R + \sqrt{\left(\frac{1}{\rho_*} + \frac{1}{\rho_R}\right)(\rho_* - \rho_R)}, \tag{A.27}$$

$$\rho_* > \rho_R, \qquad u_* > u_R. \tag{A.28}$$

- Left rarefaction:

$$u_* = u_L - 2\sqrt{2}(\sqrt{\rho_*} - \sqrt{\rho_L}), \tag{A.29}$$

$$\rho_* < \rho_L, \qquad u_* > u_L. \tag{A.30}$$

- Right rarefaction:

$$u_* = u_R + 2\sqrt{2}(\sqrt{\rho_*} - \sqrt{\rho_R}), \tag{A.31}$$

$$\rho_* < \rho_R, \qquad u_* < u_R. \tag{A.32}$$

The solution for the density ρ_* in the Star Region is given by the root of the algebraic equation:

$$f(\rho) := f_L(\rho) + f_R(\rho) + (u_R - u_L) = 0, \tag{A.33}$$

where $K = L$, R and

$$f_K(\rho) = \begin{cases} \sqrt{\left(\frac{1}{\rho_K} + \frac{1}{\rho}\right)(\rho - \rho_K)} & \text{if } \rho > \rho_K \text{ (shock)}, \\ 2\sqrt{2}(\sqrt{\rho} - \sqrt{\rho_K}) & \text{if } \rho < \rho_K \text{ (rarefaction)}. \end{cases} \tag{A.34}$$

Once we obtain the density ρ_*, the solution for velocity u_* in the Star Region is given by:

$$u_* = \frac{1}{2}\big[u_L + u_R + f_R(\rho_*) - f_L(\rho_*)\big]. \tag{A.35}$$

The numerical solution for Equation (A.33) is found by Newton-Raphson method:

$$\rho_k = \rho_{k-1} - \frac{f(\rho_{k-1})}{f'(\rho_{k-1})}, \tag{A.36}$$

where

$$f'_K(\rho) = \begin{cases} \dfrac{\frac{1}{2\rho} + \frac{\rho_K}{2\rho^2} + \frac{1}{\rho_K}}{\sqrt{\frac{1}{\rho_K} + \frac{1}{\rho}}} & \text{if } \rho > \rho_K \text{ (shock)} \ , \\ \sqrt{\frac{2}{\rho}} & \text{if } \rho < \rho_K \text{ (rarefaction)} \ . \end{cases} \tag{A.37}$$

The iteration procedure is stopped whenever the relative pressure change is less than a prescribed tolerance, for instance, $TOL = 10^{-6}$:

$$2\frac{|\rho_k - \rho_{k-1}|}{\rho_k + \rho_{k-1}} < TOL. \tag{A.38}$$

The solution inside the left rarefaction wave is sought by solving:

$$u - a = \frac{x}{t} \ , \tag{A.39}$$

$$u + 2a = u_L + 2a_L \ . \tag{A.40}$$

Through simple algebraic manipulations, we attain:

$$u = \frac{1}{3}\left(u_L + 2a_L + 2\frac{x}{t}\right) \ , \tag{A.41}$$

$$a = \frac{1}{3}\left(u_L + 2a_L - \frac{x}{t}\right) \ . \tag{A.42}$$

Using the definition of the sound speed a yields:

$$\rho = \frac{1}{18}\left(u_L + 2a_L - \frac{x}{t}\right)^2. \tag{A.43}$$

In a similar way, we obtain the solution inside the right rarefaction wave:

$$\rho = \frac{1}{18}\left(-u_R + 2a_R + \frac{x}{t}\right)^2 \ , \tag{A.44}$$

$$u = \frac{1}{3}\left(u_R - 2a_R + 2\frac{x}{t}\right) \ . \tag{A.45}$$

Let us call $s = x/t$ the speed of given particle (x, t) at the final time. We now provide a solution sampling procedure to develop a solver to find the exact solution of the complete wave structure for the Riemann problem at any point (x,t). Equations (A.7), (A.18) and (A.20) give velocities of shock and rarefaction waves:

$$(v_1; v_1^*) = \begin{cases} (u_L - a_L; u_* - a_*) & \text{if } \rho_* < \rho_L \text{ (left rarefaction)} , \\ u_L - \rho_* \sqrt{\left(\frac{1}{\rho_L} + \frac{1}{\rho_*}\right)} & \text{if } \rho_* > \rho_L \text{ (left shock)} . \end{cases} \tag{A.46}$$

$$(v_2; v_2^*) = \begin{cases} (u_R + a_R; u_* + a_*) & \text{if } \rho_* < \rho_R \text{ (right rarefaction)} , \\ u_R + \rho_* \sqrt{\left(\frac{1}{\rho_R} + \frac{1}{\rho_*}\right)} & \text{if } \rho_* > \rho_R \text{ (right shock)} . \end{cases} \tag{A.47}$$

Finally, the complete solution for the shallow water equations in case of non-vacuum reads, with $W = (\rho, u, p)$:

$$W(x,t) = \begin{cases} W_L & \text{if } s \leq v_1, \\ W_{Lfan} & \text{if } v_1 < s \leq v_1^*, \\ W_* & \text{if } v_1^* < s < v_2^*, \\ W_{Rfan} & \text{if } v_2^* \leq s < v_2, \\ W_R & \text{if } v_2 \leq s. \end{cases} \tag{A.48}$$

where ρ_{Lfan}, u_{Lfan}, ρ_{Rfan} and u_{Rfan} are given by (A.43), (A.41), (A.43) and (A.45) and the pressure is obtained by the equation of state A.1c.

A.2 Vacuum case

In a vacuum region characterised by the condition $\rho = 0$, a shock wave can not appear. This property can be obtained easily by observing the entropy condition. Therefore, we only consider the case of a two-rarefaction wave. Application of the Generalised Riemann Invariant to connect points on left and right states to a point along the contact gives:

$$v_1* = u_L + 2\sqrt{2\rho_L}, \tag{A.49}$$

$$v_2* = u_R - 2\sqrt{2\rho_R}. \tag{A.50}$$

So, we obtain the complete solution for the shallow water equations in the presence of vacuum with $W_0 = (0, u_0, 0)$:

$$W(x,t) = \begin{cases} W_L & \text{if } s \leq v_1, \\ W_{Lfan} & \text{if } v_1 < s \leq v_1^*, \\ W_0 & \text{if } v_1^* < s < v_2^*, \\ W_{Rfan} & \text{if } v_2^* \leq s < v_2, \\ W_R & \text{if } v_2 \leq s. \end{cases} \tag{A.51}$$

where u_0 is given by Equation A.35.

Appendix B

Playing with Burgers' equation

B.1 Introduction

Computer codes developed for the simulation of inviscid and non heat-conducting compressible flows are in general based on the conservative form of the Euler equations, which read in the one-dimensional case:

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{B.1a}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0, \tag{B.1b}$$

$$\partial_t E + \partial_x\big((E+p)u\big) = 0, \tag{B.1c}$$

where t stands for the time, ρ, u and p are the density, velocity and pressure in the flow, and E stands for the total energy, $E = \rho u^2/2 + \rho e$, with e the internal energy. This system must be complemented by an equation of state, giving for instance the pressure as a function of the density and the internal energy $p = \wp(\rho, e)$.

For physical reasons, the density and internal energy must be non-negative (in usual applications, positive). In addition, for the continuous problem as well as, at the discrete level, for a wide range of schemes (the so-called conservative schemes), the non-negativity of these variables allows a (weak) control on the solution; assuming that ρ and E are known on the parts of the boundary where the flow is entering the computational domain, Equations (B.1a) and (B.1c) indeed yield an $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$-estimate (with $\Omega \times (0,T)$ the space-time domain of computation) for the density and the total energy respectively. The positivity of the density at the discrete level is easily obtained from a convenient discretization of (B.1a). The positivity of the internal energy does not seem easily obtained other than by replacing Equation (B.1c) by a balance equation for the internal energy in the discrete problem; this balance equation is formally derived (*i.e.* supposing that the solution is regular) from (B.1b) and (B.1c) and reads:

$$\partial_t(\rho e) + \partial_x(\rho e u) + p\partial_x u = 0. \tag{B.2}$$

In this relation, the discrete convection operator may be built so as to respect the positivity of e: provided that the equation of state is such that for any value of ρ, p vanishes for $e = 0$, testing the discrete counterpart of (B.2) by the negative part of e proves $e \geq 0$ (see [42] for the initial chapter, [16, Appendix B] for another proof suitable in this context, and [29] in the framework of the compressible Navier-Stokes equations).

Instead of Equation (B.1c), one may also prefer to use a conservation equation for the physical entropy s, because this equation (derived for regular solutions) is a simple transport equation:

$$\partial_t(\rho s) + \partial_x(\rho s u) = 0. \tag{B.3}$$

Let us then consider that, for computational efficiency or robustness reasons, (B.2) or (B.3) are preferred to (B.1c). Since both (B.2) and (B.3) are derived from (B.1c) assuming a regular solution, there is no reason for their discretization to yield the correct weak solutions in the presence of shocks. Nevertheless, we may reasonably expect to recover the correct shock solutions if we use the following strategy:

(i) regularize the problem by adding a small diffusion term,

(ii) derive the counterpart of (B.2) or (B.3) taking into account the diffusion terms,

(iii) solve these equations,

(iv) let ϵ tend to zero.

Of course, step (iii) is performed numerically, and convergence is monitored by the space and time discretization steps h and k; the question which arises is then to find a convenient way to let ϵ and the numerical parameters h and k tend to zero. The aim of this chapter is to perform numerical experiments in order to investigate this issue on a toy problem, namely the inviscid Burgers equation. Note that we only consider explicit schemes in this study.

B.2 The equations and the numerical schemes

The inviscid Burgers equation reads:

$$\partial_t u + \partial_x(u^2) = 0, \qquad \text{for } x \in \mathbb{R},\ t \in (0,T), \tag{B.4}$$

which we complement with the initial condition:

$$u(x,0) = u_0(x), \qquad \text{for } x \in \mathbb{R}. \tag{B.5}$$

Following the above mentioned strategy (items (i)-(iv)), we first add to (B.4) a viscous term, to obtain: $\partial_t u + \partial_x(u^2) - \epsilon \partial_{xx} u = 0$. Now, multiplying this relation by $2u$ yields the following perturbed equation:

$$\partial_t u^2 + \frac{4}{3}\,\partial_x u^3 - 2u\epsilon\partial_{xx} u = 0. \tag{B.6}$$

For $\varepsilon = 0$, we get the following "Burgers square entropy" equation:

$$\partial_t u^2 + \frac{4}{3}\,\partial_x u^3 = 0. \tag{B.7}$$

which also reads, setting $v = u^2$:

$$\partial_t v + \frac{4}{3}\partial_x(v^{\frac{3}{2}}) = 0. \tag{B.8}$$

We consider the following initial data, chosen such that the entropy solution of (B.4)-(B.5) contains a discontinuity:

$$u_0(x) = \begin{cases} 10, & x \leq -0.25 \\ 1, & x > -0.25 \end{cases}. \tag{B.9}$$

It is well known that for such an initial condition, the entropy weak solutions of equations (B.4) and (B.7) differ. Let us then turn to their numerical approximations. Since the chosen initial data (B.9) is positive, the celebrated Godunov scheme reduces for both equations to the classical upwind scheme, thanks to the fact that the upwind scheme preserves (for these equations) the sign of the solution; it is well known that it leads to an approximate solution which converges, under a so called CFL condition, to the exact solution as the discretization parameters go to zero [13] (note that this is not the case for the centred finite volume scheme, although it is conservative). For the sake of simplicity, we consider constant time and space steps h and k. For $i \in \mathbb{Z}$, we set $x_i = ih$ and for $n \in \{0, \ldots, M\}$, with $(M-1)k < T \leq Mk$, we set $t_n = nk$. The discrete unknowns are the real numbers $u_i^{(n)}$, with $i \in \mathbb{Z}$ and $n \in \{0, \ldots, M\}$. The values $u_i^{(0)}$ are obtained with the initial condition:

$$u_i^{(0)} = \frac{1}{h} \int_{x_i - \frac{h}{2}}^{x_i + \frac{h}{2}} u_0(x) dx. \tag{B.10}$$

Since the discrete solution is positive, the upwind scheme for Equation (B.4) reads:

$$u_i^{(n)} = u_i^{(n-1)} + \frac{k}{h} \Big[\big(u_{i-1}^{(n-1)}\big)^2 - \big(u_i^{(n-1)}\big)^2 \Big]. \tag{B.11}$$

For this particular problem and scheme, the maximum value for the solution is reached at the initial time step so that the CFL number is the number G such that:

$$k = G \frac{h}{\max\{2s,\ s \in [1, 10]\}} = G \frac{h}{20}. \tag{B.12}$$

Similarly, the upwind scheme for Equation (B.8) reads:

$$v_i^{(n)} = v_i^{(n-1)} + \frac{4k}{3h} \Big[\big(v_{i-1}^{(n-1)}\big)^{\frac{3}{2}} - \big(v_i^{(n-1)}\big)^{\frac{3}{2}} \Big], \tag{B.13}$$

and the CFL number is the same number G. The numerical solutions obtained with (B.11) for the Burgers equation (B.4) and with (B.13) for the Burgers square entropy equation (B.7) are depicted in Figure B.1. Both are obtained with CFL equal to 1, for $T = 1/20$ and with various values of N, starting from $N = 200$ and multiplying successively by two the number of cells up to $N = 1600$. As expected, the upwind scheme (B.13) yields a numerical solution which converges (as the discretization parameters go to zero and under a CFL condition) to a weak solution of (B.7) (and even to its entropy solution), which is not a weak solution of (B.4), since the Rankine-Hugoniot conditions differ. At time $T = 1/20$, the shock for the solution of (B.4) is located at $x = 0.3$, while the shock of the solution of (B.7) is located at $x > 0.4$.

Remark B.2.1 (Link with a non-conservative diffusion term)**.** *For the Burgers equation* (B.4)*, upwinding may be seen as adding a diffusion, namely discretizing (since* $u > 0$*):*

$$\partial_t u + \partial_x(u^2) - \partial_x((hu - 2ku^2)\partial_x u) = 0.$$

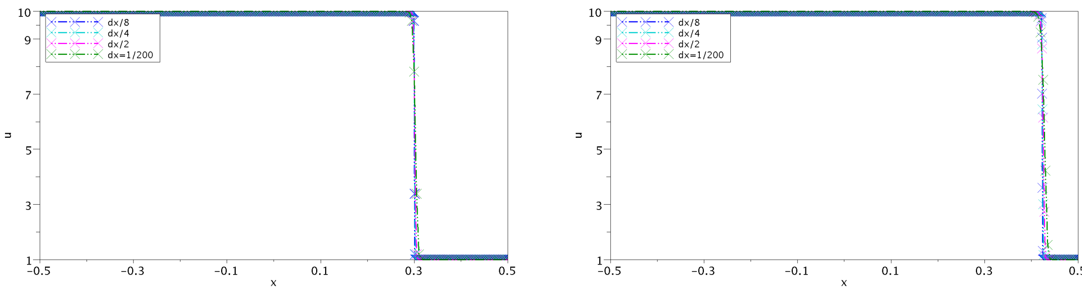


Figure B.1: Upwind Scheme for (B.4)-(B.9) (left) and (B.7)-(B.9) (right) with different mesh sizes, $CFL = 1$.

Note that one has $hu - 2ku^2 \geq 0$ *thanks to the CFL condition. For the Burgers square entropy equation* (B.7), *upwinding may be seen, formally, as solving the following parabolic equation (since* $u > 0$*):* $\partial_t u^2 + (4/3)\partial_x(u^3) - \partial_x((2hu^2 - 4ku^3)\partial_x u) = 0$. *This equation is equivalent to the following parabolic perturbation of the Burgers equation:*

$$\partial_t u + \partial_x(u^2) - \frac{1}{u}\partial_x((hu^2 - 2ku^3)\partial_x u) = 0.$$

The third term at the left-hand side may be seen as a numerical diffusion (thanks to the CFL condition) which is not in a conservative form, because of the factor $1/u$. *The above numerical results show that such a non conservative diffusion may lead to wrong discontinuities.*

B.3 Numerical solution of the perturbed equation

We then discretize the perturbed equation (B.6) with $\epsilon = \epsilon_0 h^\alpha$, where $\epsilon_0 > 0$ and $\alpha > 0$ are fixed. Note that, setting $v = u^2$, (B.6) can also be recast as:

$$\partial_t v + \frac{4}{3}\partial_x(v^{\frac{3}{2}}) - v^{\frac{1}{2}}\epsilon_0 h^\alpha \partial_x(v^{-\frac{1}{2}}\partial_x v) = 0,$$

that is a nonlinear hyperbolic equation augmented with a nonlinear nonconservative diffusion term. The upwind finite volume discretization of this equation reads (in the u

variable), with $u_i^{(0)}$ given by (B.10),

$$\big(u_i^{(n)}\big)^2 = \big(u_i^{(n-1)}\big)^2 + \frac{4k}{3h}\Big[\big(u_{i-1}^{(n-1)}\big)^3 - \big(u_i^{(n-1)}\big)^3\Big] + \frac{k}{h^2}\,\epsilon_0 h^\alpha\, u_i^{(n-1)}\Big[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\Big]. \tag{B.14}$$

We present in Figures B.2, B.3 and B.4 the numerical solutions obtained with (B.14) for $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 2$ respectively, and for the same time $T = 1/20$, CFL=0.1 and meshes as in Section B.2. The parameter ϵ_0 is such that $\epsilon_0 h^\alpha = 0.2$ for $N = 200$ (whatever α may be). Figure B.2 shows that for $0 < \alpha < 1$, the sequence of approximate solutions given by (B.14) converges to a weak solution of the initial Burgers equation (B.4), as h and k tend to 0, under a stability condition, which, since $\alpha < 1$, becomes more stringent than a CFL condition when h tends to zero. Figure B.3 shows that for $\alpha > 1$, we obtain the convergence to the solution of (B.7); figure B.4 shows that for $\alpha = 1$, the location of the discontinuity lies in between the discontinuities of the solution to (B.6) and (B.7). These results seem to indicate that the convergence to the solution of (B.7) (resp. (B.6)) occurs when the added diffusion dominates (resp. is dominated by) the numerical one.

Let us finally study the following finite volume centred scheme for Equation (B.7), which reads:

$$\big(u_i^{(n)}\big)^2 = \big(u_i^{(n-1)}\big)^2 + \frac{4k}{3h}\Big[\big(\frac{u_{i-1}^{(n-1)} + u_i^{(n-1)}}{2}\big)^3 - \big(\frac{u_i^{(n-1)} + u_{i+1}^{(n-1)}}{2}\big)^3\Big] + \frac{k}{h^2}\,\epsilon_0 h^\alpha\, u_i^{(n-1)}\Big[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\Big]. \tag{B.15}$$

Results for $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ (and ϵ_0 such that $\epsilon_0 h^\alpha = 0.2$ for $N = 200$, whatever α may be) are reported on Figures B.5, B.6 and B.7, respectively. The numerical solution now seems to converge to the solution of (B.7), at least for $\alpha \in (0, 2)$. For the finest mesh and $\alpha = 2$, the diffusion is no longer sufficient to prevent some spurious oscillations near the shock. Last but not least, the additional diffusion which is necessary to recover the right shock location is considerably reduced with respect to the upwind scheme (even if the scheme still appears more diffusive than the standard upwind scheme applied to (B.4)), which is encouraging in view of practical extensions to Euler equations.

Conclusion We tested two discretizations for the modified equation (B.6):

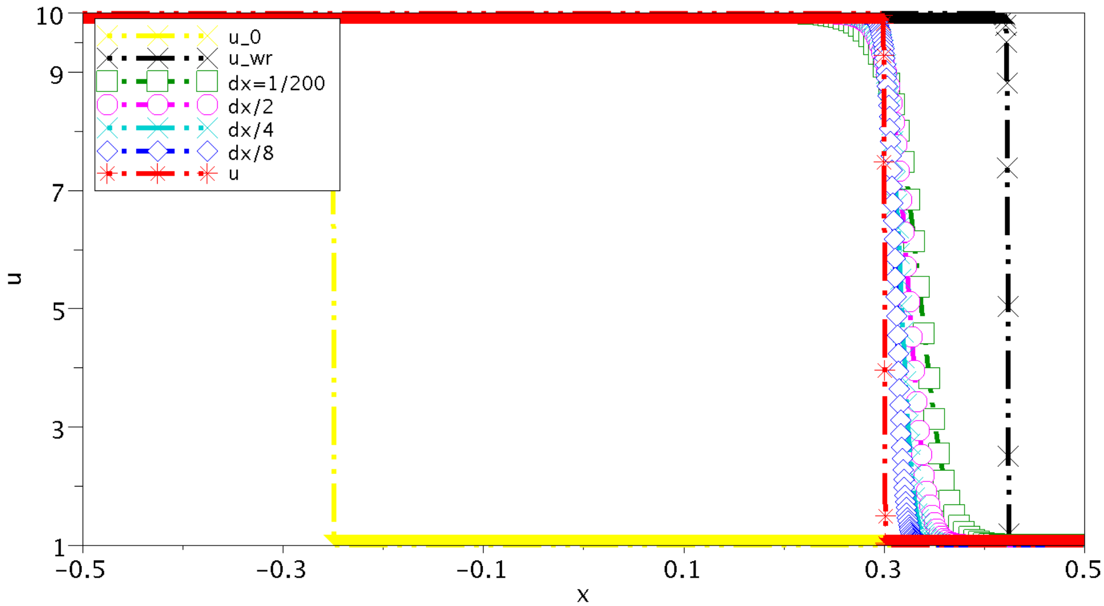


Figure B.2: Upwind Scheme for (B.6) with non conservative diffusion term, $\alpha = 0.5$.

– an upwind scheme for which the solution converges to the weak solution of (B.4) if the viscous term is predominant with respect to the numerical diffusion, that is if $\epsilon = \epsilon_0 h^\alpha$, with $\epsilon_0 > 0$ and $\alpha \in (0, 1)$.

– a centred scheme which yields correct solutions for all values $\alpha \in (0, 2)$.

The extension of this work to Euler equations is under way, and results are encouraging. Indeed, it seems that we are able to build convergent schemes, even in the presence of shocks, using either the entropy or internal energy balance. A next step might be to use a nonlinear viscosity to avoid an excessive smearing of the solutions, following the ideas developed in [22].

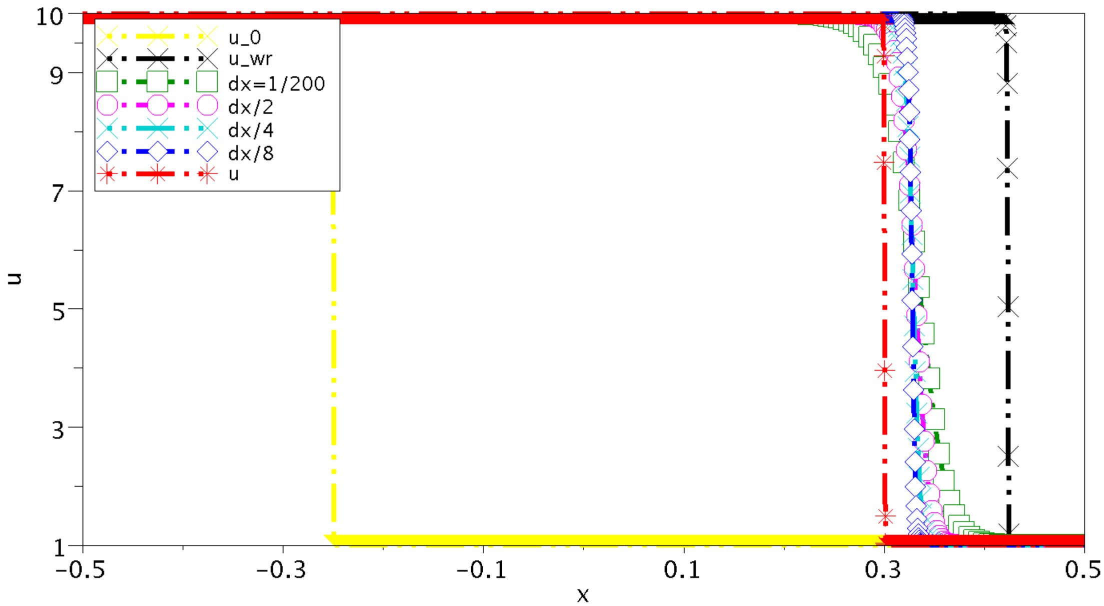


Figure B.3: Upwind Scheme for (B.6) with non conservative diffusion term, $\alpha = 1$.

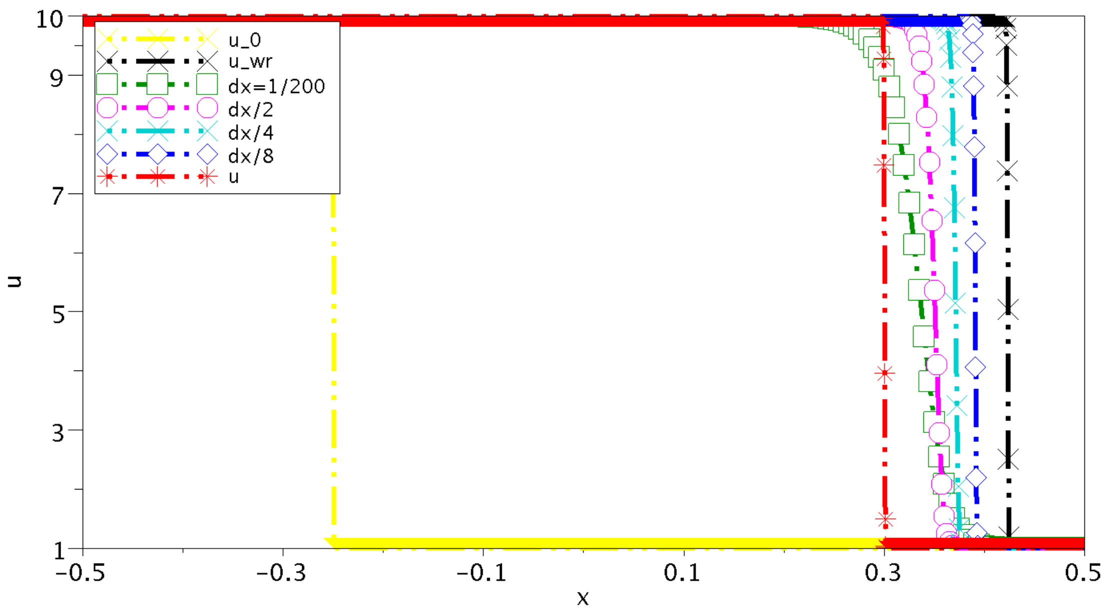


Figure B.4: Upwind Scheme for (B.6) with non conservative diffusion term, $\alpha = 2$.

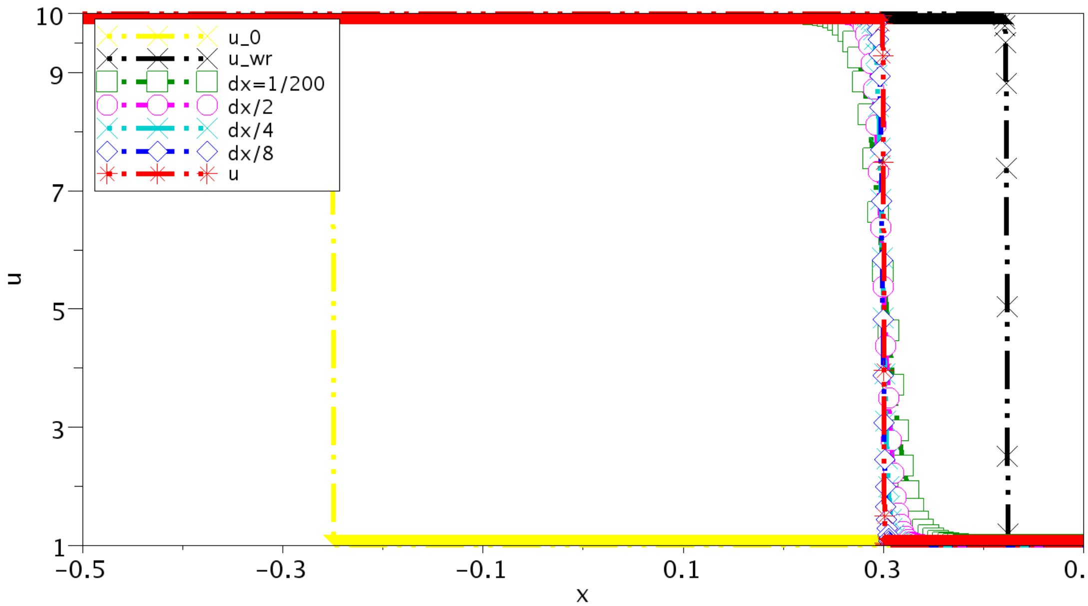


Figure B.5: Centered Scheme for (B.6) with non conservative diffusion term, $\alpha = 1$.

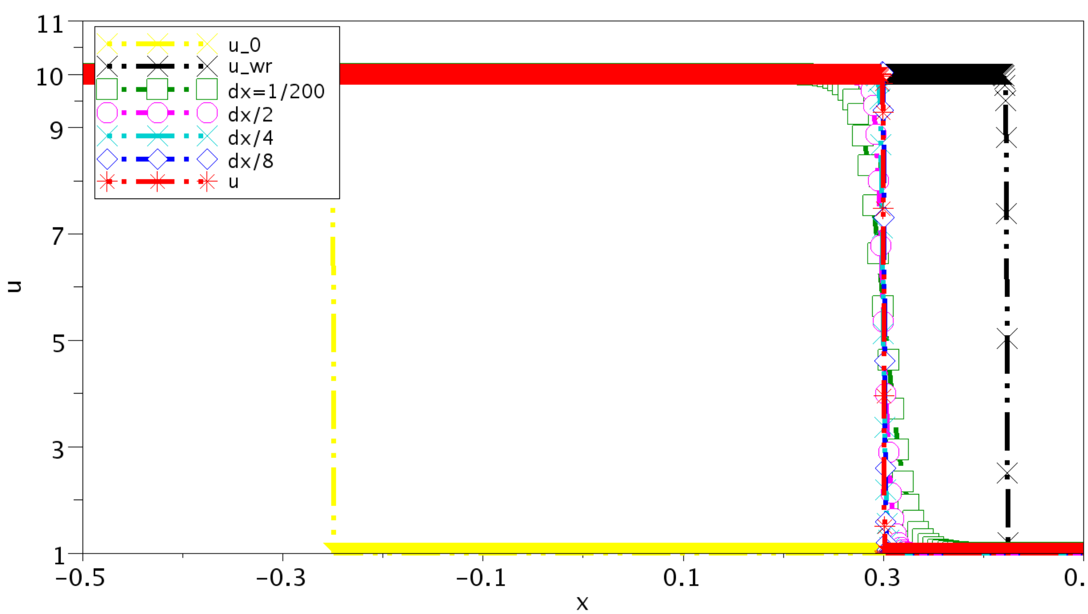


Figure B.6: Centered Scheme for (B.6) with non conservative diffusion term, $\alpha = 1.5$.

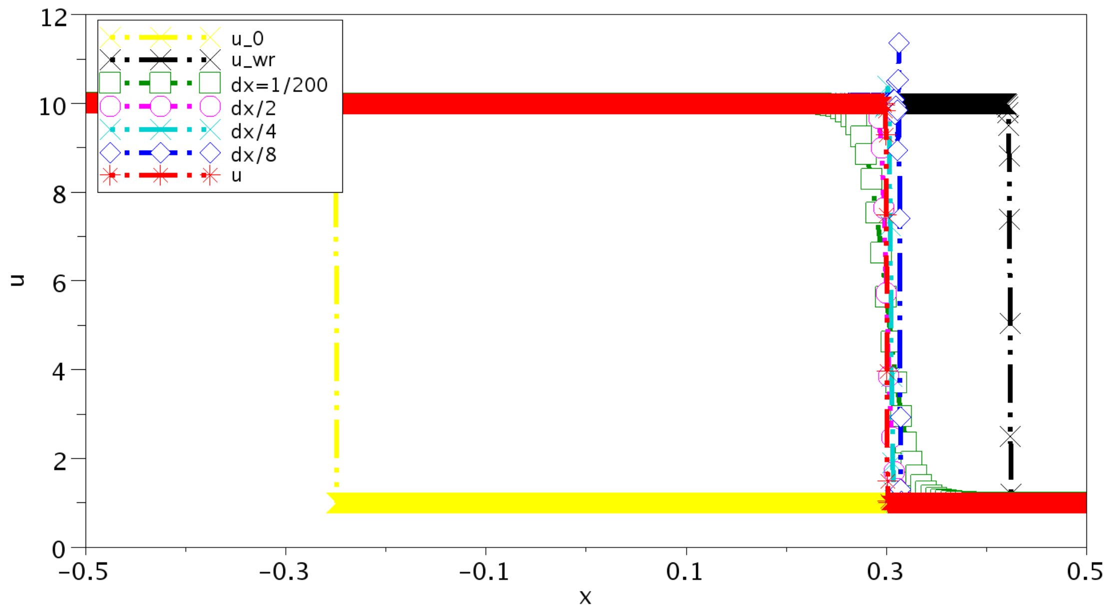


Figure B.7: Centered Scheme for (B.6) with non conservative diffusion term, $\alpha = 2$.

Bibliography

[1] G. Ansanay-Alex, F. Babik, J.-C. Latché, and D. Vola. An L^2-stable approximation of the Navier-Stokes convection operator for low-order non-conforming finite elements. *International Journal for Numerical Methods in Fluids*, 66:555–580, 2011.

[2] F. Archambeau, J.-M. Hérard, and J. Laviéville. Comparative study of pressure-correction and Godunov-type schemes on unsteady compressible cases. *Computers & Fluids*, 38:1495–1509, 2009.

[3] R. Babik, R. Herbin, W. Kheriji, and J.-C. Latché. Discretization of the viscous dissipation term with the mac scheme. In *Finite Volumes for Complex Applications VI - Problems and Perspectives - Prague, Czech Republic*, 2011.

[4] R.A. Berry. Notes on PCICE method: simplification, generalization and compressibility properties. *Journal of Computational Physics*, 215:6–11, 2006.

[5] H. Bijl and P. Wesseling. A unified method for computing incompressible and compressible flows in boundary-fitted coordinates. *Journal of Computational Physics*, 141:153–173, 1998.

[6] F. Bouchut. *Nonlinear Stability of finite volume methods for hyperbolic conservation laws*. Birkhauser, 2004.

[7] E.J. Caramana, D.E. Burton, M.J. Shashkov, and P.P. Whalen. The construction of compatible hydrodynamics algorithms utilizing conservation of total energy. *Journal of Computational Physics*, 146:227–262, 1998.

[8] V. Casulli and D. Greenspan. Pressure method for the numerical solution of transient, compressible fluid flows. *International Journal for Numerical Methods in Fluids*, 4:1001–1012, 1984.

[9] A.J. Chorin. Numerical solution of the Navier-Stokes equations. *Mathematics of Computation*, 22:745–762, 1968.

[10] P. G. Ciarlet. Basic error estimates for elliptic problems. In P. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, Volume II*, pages 17–351. North Holland, 1991.

[11] M. Crouzeix and P.A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO Série Rouge*, 7:33–75, 1973.

[12] I. Demirdžić, Ž. Lilek, and M. Perić. A collocated finite volume method for predicting flows at all speeds. *International Journal for Numerical Methods in Fluids*, 16:1029–1050, 1993.

[13] R. Eymard, T. Gallouët, and R. Herbin. *Finite Volume Methods*. North Holland, 2000.

[14] R. Eymard, T. Gallouët, R. Herbin, and J.-C. Latché. Convergence of the MAC scheme for the compressible Stokes equations. *SIAM Journal on Numerical Analysis*, 48:2218–2246, 2010.

[15] T. Gallouët, L. Gastaldo, R. Herbin, and J.-C. Latché. An unconditionally stable pressure correction scheme for compressible barotropic Navier-Stokes equations. *Mathematical Modelling and Numerical Analysis*, 42:303–331, 2008.

[16] T. Gallouët, A. Larcher, and J.-C. Latché. Convergence of a finite volume scheme for the convection-diffusion equation with l^1 data. *Mathematics of Computation*, 2011.

[17] L. Gastaldo, R. Herbin, W. Kheriji, C. Lapuerta, and J.-C. Latché. Staggered discretizations, pressure correction schemes and all speed barotropic flows. In *Finite Volumes for Complex Applications VI - Problems and Perspectives - Prague, Czech Republic*, volume 2, pages 39–56, 2011.

[18] L. Gastaldo, R. Herbin, and J.-C. Latché. A discretization of phase mass balance in fractional step algorithms for the drift-flux model. *IMA Journal of Numerical Analysis*, 31:116–146, 2011.

[19] E. Godlewski and P.-A. Raviart. *Numerical approximation of hyperbolic systems of conservation laws*. Springer, 1996.

[20] J.L. Guermond, P. Minev, and J. Shen. An overview of projection methods for incompressible flows. *Computer Methods in Applied Mechanics and Engineering*, 195:6011–6045, 2006.

[21] J.L Guermond and R. Pasquetti. Entropy-based nonlinear viscosity for Fourier approximations of conservation laws. *Comptes Rendus de l'Académie des Sciences de Paris – Série I – Analyse Numérique*, 346:801–806, 2008.

[22] J.L Guermond, R. Pasquetti, and B. Popov. Entropy viscosity method for nonlinear conservation laws. *Journal of Computational Physics*, 230:4248–4267, 2011.

[23] F.H. Harlow and A.A. Amsden. Numerical calculation of almost incompressible flow. *Journal of Computational Physics*, 3:80–93, 1968.

[24] F.H. Harlow and A.A. Amsden. A numerical fluid dynamics calculation method for all flow speeds. *Journal of Computational Physics*, 8:197–213, 1971.

[25] F.H. Harlow and J.E. Welsh. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids*, 8:2182–2189, 1965.

[26] R. Herbin, W. Kheriji, and J.-C. Latché. Consistent semi-implicit staggered schemes for compressible flows - Part I: Barotropic Euler equations. *submitted*, 2012.

[27] R. Herbin, W. Kheriji, and J.-C. Latché. Consistent semi-implicit staggered schemes for compressible flows - Part II: Euler equations. *submitted*, 2012.

[28] R. Herbin, W. Kheriji, and J.-C. Latché. Staggered schemes for all speed flows. *Esaim Proc.*, 2012.

[29] R. Herbin, W. Kheriji, and J.-C. Latché. An unconditionally stable pressure correction scheme for compressible Navier-Stokes equations. *submitted*, 2012.

[30] R. Herbin and J.-C. Latché. Kinetic energy control in the MAC discretization of the compressible Navier-Stokes equations. *International Journal of Finites Volumes*, 7, 2010.

[31] B.H. Hjertager. Computer simulation of reactive gas dynamics. *Modeling, Identification and Control*, 5:211–236, 1985.

[32] Y. Hou and K. Mahesh. A robust, colocated, implicit algorithm for direct numerical simulation of compressible, turbulent flows. *Journal of Computational Physics*, 205:205–221, 2005.

[33] ISIS. A CFD computer code for the simulation of reactive turbulent flows. `https://gforge.irsn.fr/gf/project/isis`.

[34] R.I. Issa. Solution of the implicitly discretised fluid flow equations by operator splitting. *Journal of Computational Physics*, 62:40–65, 1985.

[35] R.I. Issa, A.D. Gosman, and A.P. Watkins. The computation of compressible and incompressible recirculating flows by a non-iterative implicit scheme. *Journal of Computational Physics*, 62:66–82, 1986.

[36] R.I. Issa and M.H. Javareshkian. Pressure-based compressible calculation method utilizing total variation diminishing schemes. *AIAA Journal*, 36:1652–1657, 1998.

[37] S.Y. Kadioglu, M. Sussman, S. Osher, J.P. Wright, and M. Kang. A second order primitive preconditioner for solving all speed multi-phase flows. *Journal of Computational Physics*, 209:477–503, 2005.

[38] K.C. Karki and S.V. Patankar. Pressure based calculation procedure for viscous flows at all speeds in arbitrary configurations. *AIAA Journal*, 27:1167–1174, 1989.

[39] W. Kheriji, R. Herbin, and J.-C. Latché. Pressure correction staggered schemes for barotropic monophasic and two-phase flows. *submitted*, 2012.

[40] M.H. Kobayashi and J.C.F. Pereira. Characteristic-based pressure correction at all speeds. *AIAA Journal*, 34:272–280, 1996.

[41] N. Kwatra, J. Su, J.T. Grétarsson, and R. Fedkiw. A method for avoiding the acoustic time step restriction in compressible flow. *Journal of Computational Physics*, 228:4146–4161, 2009.

[42] B. Larrouturou. How to preserve the mass fractions positivity when computing compressible multi-component flows. *Journal of Computational Physics*, 95:59–84, 1991.

[43] F.-S. Lien. A pressure-based unstructured grid method for all-speed flows. *International Journal for Numerical Methods in Fluids*, 33:355–374, 2000.

[44] M.-S. Liou. A sequel to AUSM, part II:AUSM+-up. *Journal of Computational Physics*, 214:137–170, 2006.

[45] M.-S. Liou and C.J. Steffen. A new flux splitting scheme. *Journal of Computational Physics*, 107:23–39, 1993.

[46] A. Majda and J. Sethian. The derivation and numerical solution of the equations for zero Mach number solution. *Combustion Science and Techniques*, 42:185–205, 1985.

[47] R.C. Martineau and R.A. Berry. The pressure-corrected ICE finite element method for compressible flows on unstructured meshes. *Journal of Computational Physics*, 198:659–685, 2004.

[48] J.J. McGuirk and G.J. Page. Shock capturing using a pressure-correction method. *AIAA Journal*, 28:1751–1757, 1990.

[49] F. Moukalled and M. Darwish. A high-resolution pressure-based algorithm for fluid flow at all speeds. *Journal of Computational Physics*, 168:101–133, 2001.

[50] V. Moureau, C. Bérat, and H. Pitsch. An efficient semi-implicit compressible solver for large-eddy simulations. *Journal of Computational Physics*, 226:1256–1270, 2007.

[51] K. Nerinckx, J. Vierendeels, and E. Dick. Mach-uniformity through the coupled pressure and temperature correction algorithm. *Journal of Computational Physics*, 206:597–623, 2005.

[52] K. Nerinckx, J. Vierendeels, and E. Dick. A Mach-uniform algorithm: coupled versus segregated approach. *Journal of Computational Physics,* 224:314–331, 2007.

[53] P. Nithiarasu, R. Codina, and O.C. Zienkiewicz. The Characteristic-Based Split (CBS) scheme – A unified approach to fluid dynamics. *International Journal for Numerical Methods in Engineering*, 66:1514–1546, 2006.

[54] G. Patnaik, R.H. Guirguis, J.P. Boris, and E.S. Oran. A barely implicit correction for flux-corrected transport. *Journal of Computational Physics*, 71:1–20, 1987.

[55] PELICANS. Collaborative development environment. `https://gforge.irsn.fr/gf/project/pelicans`.

[56] E.S. Politis and K.C. Giannakoglou. A pressure-based algorithm for high-speed turbomachinery flows. *International Journal for Numerical Methods in Fluids*, 25:63–80, 1997.

[57] R. Rannacher and S. Turek. Simple nonconforming quadrilateral Stokes element. *Numerical Methods for Partial Differential Equations*, 8:97–111, 1992.

[58] E.A. Sewall and D.K. Tafti. A time-accurate variable property algorithm for calculating flows with large temperature variations. *Computers & Fluids*, 37:51–63, 2008.

[59] R. Temam. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires II. *Arch. Rat. Mech. Anal.*, 33:377–385, 1969.

[60] S. Thakur and J. Wright. A multiblock operator-splitting algorithm for unsteady flows at all speeds in complex geometries. *International Journal for Numerical Methods in Fluids*, 46:383–413, 2004.

[61] E. Toro. *Riemann solvers and numerical methods for fluid dynamics – A practical introduction (third edition)*. Springer, 2009.

[62] D.R. Van der Heul, C. Vuik, and P. Wesseling. Stability analysis of segregated solution methods for compressible flow. *Applied Numerical Mathematics*, 38:257–274, 2001.

[63] D.R. Van der Heul, C. Vuik, and P. Wesseling. A conservative pressure-correction method for flow at all speeds. *Computers & Fluids*, 32:1113–1132, 2003.

[64] J.P. Van Dormaal, G.D. Raithby, and B.H. McDonald. The segregated approach to predicting viscous compressible fluid flows. *Transactions of the ASME*, 109:268–277, 1987.

[65] D. Vidović, A. Segal, and P. Wesseling. A superlinearly convergent Mach-uniform finite volume method for the Euler equations on staggered unstructured grids. *Journal of Computational Physics*, 217:277–294, 2006.

[66] C. Wall, C.D. Pierce, and P. Moin. A semi-implicit method for resolution of acoustic waves in low Mach number flows. *Journal of Computational Physics*, 181:545–563, 2002.

[67] I. Wenneker, A. Segal, and P. Wesseling. A Mach-uniform unstructured staggered grid method. *International Journal for Numerical Methods in Fluids*, 40:1209–1235, 2002.

[68] P. Wesseling. Principles of computational fluid dynamics. volume 29 of *Springer Series in Computational Mathematics*. Springer, 2001.

[69] C.M. Xisto, J.C. Páscoa, P.J. Oliveira, and D.A. Nicolini. A hybrid pressure-density-based algorithm for the Euler equations at all Mach number regimes. *International Journal for Numerical Methods in Fluids, online*, 2011.

[70] S.Y. Yoon and T. Yabe. The unified simulation for incompressible and compressible flow by the predictor-corrector scheme based on the CIP method. *Computer Physics Communications*, 119:149–158, 1999.

[71] O.C. Zienkiewicz and R. Codina. A general algorithm for compressible and incompressible flow – Part I. The split characteristic-based scheme. *International Journal for Numerical Methods in Fluids*, 20:869–885, 1995.

Explicit staggered schemes for compressible flows

Abstract – We develop and analyse explicit-in-time schemes for the computation of compressible flows, based on staggered in space unstructured discretization. Upwinding is performed equation by equation only with respect to the velocity (like in the AUSM family of schemes). The pressure gradient is built as the transpose of the natural divergence, which yields a centered discretization of this term.

In a first time, we address the barotropic Euler equations. The velocity convection term is built in such a way that we are able to derive a discrete kinetic energy balance, with (at the left-hand side) residual terms which are non-negative under a CFL condition. We then show that, in one space dimension, the scheme is consistent in the sense that, if a sequence of discrete solutions converges to some limit, then this limit is a weak entropy solution to the continuous problem. Numerical tests allow to check the convergence of the scheme, and show in addition an approximatively first-order convergence rate.

We then turn to the full (*i.e.* non-barotropic) Euler equations. We chose here to solve the internal energy balance instead of the total energy equation, which presents two advantages: first, we don't need a discretization of this latter quantity, which is rather unnatural since the velocity and the scalar unknowns are not approximated on the same mesh; second, an *ad hoc* discretization of the internal energy balance ensures its positivity. We show that, under CFL-like conditions, the density and internal energy are kept positive, and the total (*i.e.* integrated over the whole computational domain) energy cannot grow. The difficult point is to obtain consistency. Indeed, a scheme using the internal energy equation may not converge to a weak solution of the original system in the presence of shocks. This problem is healed by the following strategy:

1. Establish a kinetic energy identity at the discrete level (with some source terms).
2. Choose source term of the internal energy equation such that the total energy balance is recovered when the mesh and time steps tend to zero.

More precisely speaking, we prove the following theoretical result. In 1D, if we assume the L^∞ and BV-stability and the convergence of the scheme, passing to the limit of the discrete kinetic and discrete elastic potential equations, we show that the limit of the sequence of solutions indeed is a weak solution. This result is supported by numerical tests.

Finally, we consider the computation of radial flows, governed by Euler equations in axisymetrical (2D) or spherical (3D) coordinates, and obtain similar results to the previous sections.

Schémas numériques explicites pour le calcul d'écoulements compressibles

Résumé – On étudie des schémas de type explicite en temps sur maillage décalé non structuré pour l'approximation des écoulements compressibles. Pour chacune des équations considérées, séparément, un décentrement amont est effectué sur la vitesse matérielle. L'opérateur de gradient de pression discret est défini comme la transposée de l'opérateur de divergence discrète, et c'est donc un opérateur centré.

Dans un premier temps, on s'intéresse aux équations d'Euler barotrope. Le terme de convection non linéaire en vitesse est construit de manière à ce que les solution approchées satisfassent, sous condition de CFL, un bilan d'énergie cinétique discret (avec, au premier membre, un terme résiduel positif). On montre ensuite qu'en une dimension d'espace (1D), le schéma est consistant, au sens où si les solutions approchées convergent vers une limite lorsque les pas de temps et maillage tendent vers 0, alors cette limite est solution faible entropique du problème continu. Des tests numériques permettent de vérifier la convergence du schéma, avec un ordre proche de un.

Dans un deuxième temps, on traite les équations d'Euler complètes. Plutôt que de résoudre l'équation d'énergie totale, choix traditionnel des schémas colocalisés, on préfère résoudre l'équation d'énergie interne, ce qui présente deux avantages : d'une part, on évite d'avoir à discrétiser l'énergie totale, qui fait intervenir l'énergie interne et la pression, variables qui ne sont pas définies sur le même maillage ; d'autre part, une discrétisation *ad hoc* de l'énergie interne assure la positivité de cette dernière sous condition de CFL. Cependant, l'utilisation de l'équation d'énergie interne nécessite des précautions : le fait de ne pas travailler sur l'énergie totale peut en effet faire apparaître des solutions approchées qui ne tendent pas vers une solution faible des équations d'Euler, et qui en particulier, ne vérifient pas les relations de Rankine et Hugoniot et font apparaître des mauvaises vitesses de choc. Le remède est de s'assurer que le bilan d'énergie total soit bien assuré à la limite, en écrivant ce bilan comme la somme du bilan d'énergie interne et du bilan d'énergie cinétique, et en introduisant dans l'équation d'énergie interne discrète un terme de correction qui compense le terme résiduel (positif) du bilan d'énergie cinétique décrit plus haut, et qui ne tend pas vers 0. Dans ce cas encore, on montre que dans le cas 1D, si les solutions approchées convergent, alors elles convergent vers une solution faible des équations d'Euler. Les résultats numériques corroborent la théorie.

Enfin, dans une troisième partie, pour des écoulements radiaux uniquement, on discrétise des équations d'Euler en coordonnées cylindriques (2D) ou sphériques (3D); les résultats obtenus sont similaires aux précédents.